Salient features of fluid turbulence.

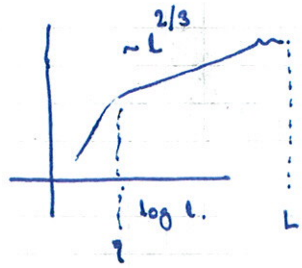
Experimental aspects:

- Structure function

$$\delta u_{\parallel}(\ell) = \left[u(\underline{x}+\ell) - u(\underline{x})\right] \cdot \frac{\underline{\ell}}{\ell}$$

$$S_p(\ell) = \left\langle \left[\delta u_{\parallel}(\ell)\right]^p \right\rangle$$

- Scaling of $S_2(\ell)$



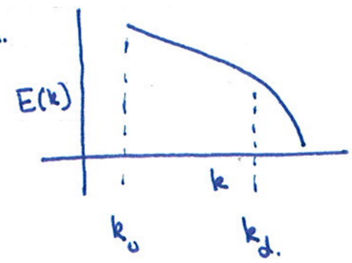
$u(x+\ell) - u(x) \sim \# \, \ell.$

for $\ell < \eta$

velocity differences are "smooth" at small scales.

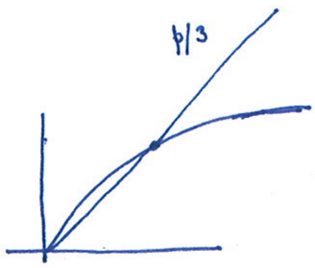
Here ~~is~~ "smooth" ≡ expandable in a Taylor series.

- Energy spectra.



$k_0 \sim \frac{1}{L}$, $k_d \sim \frac{1}{\eta}$.

- Multiscaling. $S_p(\ell) \sim \ell^{\zeta_p}$ with



$\cancel{S_p}$ $S_3(\ell) \sim \ell$ — Karman - Howarth.

- Zeroth law of turbulence. $E = \left\langle \frac{1}{2} |\underline{u}|^2 \right\rangle = \frac{1}{V} \frac{1}{2} \int \underline{u}^2(\underline{x}) \, d^3x$

$$\varepsilon \equiv - \frac{dE}{dt}$$

$$\varepsilon = -2\nu\Omega = -2\nu\left\langle |\underline{\nabla}\times\underline{u}|^2 \right\rangle$$

$$= -2\nu\int\left[\underline{\nabla}\times\underline{u}(\underline{x})\right]^2\frac{d^3x}{V}$$

As $\nu \to 0$, $\varepsilon(\nu) \to \varepsilon > 0$.

" ~~Diss~~

As ~~&~~ ε becomes smaller and smaller ~~the~~ ~~a~~ ~~curl~~ velocity becomes ~~cuig~~ more and more irregular such that the product tends to a finite positive number.

"Dissipative Anomaly"

This is a basic hypothesis behind Kolmogorov's theory of turbulence and all subsequent phenomenological theories of turbulence.

~~We shall see how different aspects of this~~ basic

~~We shall see how diff.~~

" To see the world in a grain of sand
and heaven in a wild flower "

Willam Blake

~~Positive~~ s ~~v~~

Auguries of Innosence.

An important goal of statistical theories of turbulence is to understand these features of turbulence; particularly ~~its~~ multiscaling.

Kraichnan model of passive scalar.

$$\partial_t \theta + (\underline{u} \cdot \underline{\nabla}) \theta = \varkappa \nabla^2 \theta + f^\theta .$$

~~The structure fn~~

where the velocity $\underline{u}$ is Gaussian. and white-in-time.

such that

$$\langle u^\alpha \rangle = 0$$

$$\langle u^\alpha(\underline{x}, t)\, u^\beta(\underline{x}', t') \rangle = D^{\alpha\beta}(\underline{x}, x')\, \delta(t - t')$$

Or in Fourier space.

$$\hat{u}_\alpha(\underline{k}, t) = \int u_\alpha(\underline{x}, t)\, e^{i\, \underline{k} \cdot \underline{x}}\, d^3x$$

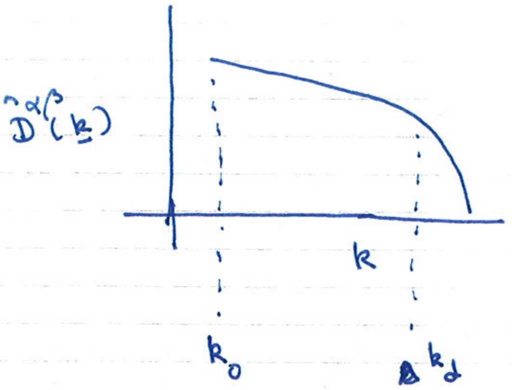
$$u_\alpha(\underline{x}, t) = \frac{1}{2\pi} \int \hat{u}_\alpha(\underline{k}, t)\, e^{-i\, \underline{k} \cdot \underline{x}}\, d^3k$$

And $$\langle \hat{u}_\alpha(\underline{k}, t)\, \hat{u}_\beta(\underline{k}', t') \rangle = \tilde{D}^{\alpha\beta}(\underline{k}, \underline{k}')\, \delta(t - t')$$

Translational invariance : $$D^{\alpha\beta}(\underline{x}, \underline{x}') = D^{\alpha\beta}(\underline{x} - \underline{x}')$$

$$\Rightarrow \quad \tilde{D}^{\alpha\beta}(\underline{k}, \underline{k}') = \hat{D}^{\alpha\beta}(\underline{k})$$

$$\hat{D}^{\alpha\beta}(\underline{k}) = \int D^{\alpha\beta}(\underline{r})\, e^{i\, \underline{k} \cdot \underline{r}}\, d^3r$$



$$\hat{D}^{\alpha\beta}(\underline{k}) = \underset{\substack{\uparrow \\ \text{incompressibility.}}}{P^{\alpha\beta}(\underline{k})} \frac{e^{-(\eta k)^2}}{\left(k^2 + k_0^2\right)^{\frac{d+\xi}{2}}}$$

with $0 \leq \xi \leq 2$

$$k_0 \sim \frac{1}{L}, \quad \eta \sim \frac{1}{k_d}$$

$$\hat{D}^{\alpha\beta}(k) \sim k^{-(d+\xi)} \qquad k_d \gg k \gg k_0$$

$$\xi \cancel{=}. \quad P^{\alpha\beta}(\underline{k}) = \delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}$$

- This mimics the turbulent velocity field.

But NOT in the following way

- white-in-time ~~is~~ a
- Gaussian nature of ~~the~~ PDF of $\underline{u}$.

The main point in the study of the whole model is that although the statistics of $\underline{u}$ is Gaussian; we shall obtain a non-Gaussian; multiscaling statistics for the passive scalar field. The statistical properties of the scalar will mimic that of turbulent velocity ~~fie~~ field.

Karman-Howarth.

$$\langle (\delta u_{\parallel})^3 \rangle = -\frac{4}{5} \epsilon r \qquad \langle \delta u_{\parallel} (\delta\theta)^2 \rangle = -\frac{4}{3} \epsilon^\theta r$$

$$\epsilon = \nu \langle (\nabla u)^2 \rangle \qquad \epsilon^\theta = \varkappa \langle (\nabla \theta)^2 \rangle$$

$$\cancel{\partial_t\theta + \underline{u}\cdot\nabla} \quad \partial_t\theta + u_\alpha\partial_\alpha\theta = \kappa\partial_{\beta\beta}\theta + f_\alpha \cancel{\theta}$$

~~We begin by first studying the str~~

We begin by first studying the ~~properties of the~~ structure function of passive scalar.

$$\cancel{S_2(t)}\theta \quad S_2(\underline{r},t) = \langle [\theta(\underline{x}+\underline{l},t) - \theta(\underline{x},t)]^2 \rangle$$

$$= 2[\langle \theta^2(\underline{x},t)\rangle - \langle \theta(\underline{x}+\underline{l},t)\,\theta(\underline{x},t)\rangle]$$

$$= 2[C_2(0,t) - C_2(\underline{r},t)]$$

$$C_2(\underline{r},t) = \langle \theta(\underline{x}+\underline{l},t)\,\theta(\underline{x},t)\rangle$$

~~The~~ ~~wa~~

$$\partial_t C_2(\underline{r},t) = \langle [\partial_t \theta(\underline{x}+\underline{l},t)]\,\theta(\underline{x},t)\rangle + \langle \theta(\underline{x}+\underline{l},t)\,\partial_t[\theta(\underline{x},t)]\rangle$$

$$= \langle (\partial_t\theta')\theta\rangle + \langle \theta\,\partial_t\theta'\rangle$$

$$\langle(\partial_t\theta')\theta\rangle$$

$$= \langle [-u'_\alpha\partial'_\alpha\theta' + \kappa\partial'_{\beta\beta}\theta' + f'_\alpha]\,\theta\rangle$$

$$= -\langle [u'_\alpha\partial'_\alpha\theta']\theta\rangle + \kappa\langle\theta\,\partial'_{\beta\beta}\theta'\rangle + \langle f'_\alpha\theta\rangle$$

$$= -\partial'_\alpha\langle u'_\alpha\theta'\theta\rangle + \kappa\,\partial'_{\beta\beta}\langle\theta\theta'\rangle + \langle f'_\alpha\theta\rangle$$

$$= -\partial'_\alpha\langle u'_\alpha\theta'\theta\rangle + \kappa\,\cancel{C_2(r,t)}\;\kappa\partial'_{\beta\beta}C_2(\underline{x},\underline{x}',t) + \langle f'\theta\rangle$$

Here $\langle\ \rangle$ denotes averaging over both the velocity and the force. We already know the statistics of θ, u; we now specify the statistics of f.

f is Gaussian, white-in-time, zero mean

$$\langle f(\underline{x}, t) \rangle = 0$$

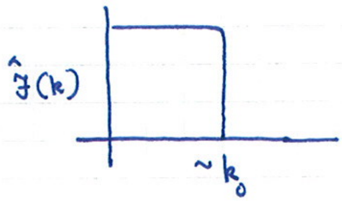
and

$$\langle f(\underline{x}, t)\, f(\underline{x}', t') \rangle = \underbrace{\mathcal{F}(\underline{x} - \underline{x}')}_{\to \text{ translational invariance.}}\, \delta(t - t')$$

and $\cancel{\hat{\mathcal{F}}(\underline{x} \to}$ $\quad \hat{\mathcal{F}}(\underline{k}) = \int \mathcal{F}(\underline{r})\, e^{i \underline{k}\cdot\underline{r}}\, d^3 r$

$$\hat{\mathcal{F}}(\underline{k}) = \hat{\mathcal{F}}(k)$$

– isotropic forcing.



consider.
$$\langle u'_\alpha \theta' \theta \rangle = \langle u'_\alpha\, \theta'[u,f]\, \theta[u,f] \rangle$$
$$= \cancel{u'_\alpha}\Lambda = \langle u'_\alpha\, A[u,f] \rangle$$

Claim :

$$\langle u'_\alpha\, A[u,f] \rangle = D^{\alpha\beta}\cancel{(\underline{x}'' - \underline{x}', t'' - t')}$$
$$= \langle u'_\alpha u''_\beta \rangle \left\langle \frac{\delta A}{\delta u''_\beta} \right\rangle$$

Proof

Let x be a Gaussian random variable. And $f(x)$ be some function of x. ~~Then~~.

$$p(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-\frac{x^2}{2\sigma^2}}$$

$$\langle x f(x) \rangle = \int_{-\infty}^{+\infty} x f(x) p(x)\, dx$$

$$= \langle x^2 \rangle \langle f'(x) \rangle$$

Now for $\underline{x} = (x_1, x_2, \ldots x_N)$

Then $\langle x_i f(\{x_i\}) \rangle = \langle x_i x_j \rangle \langle \frac{\partial f}{\partial x_j} \rangle$

Generalising to functions

$$\langle u(\underline{x}) f[u] \rangle = \langle u(\underline{x}) u(\underline{y}) \rangle \langle \frac{\delta f}{\delta u(y)} \rangle$$

Then $\langle u_\alpha(\underline{x}', t') A[u, f] \rangle = \langle u_\alpha(x', t') u_\beta(x'', t'') \rangle \langle \frac{\delta A}{\delta u_\beta(x'', t'')} \rangle$

Then $\langle u_\alpha(\underline{x}', t') \, \theta(\underline{x}', t') \, \theta(\underline{x}, t) \rangle$

$$= \langle u_\alpha(\underline{x}', t) \, u_\beta(x'', t'') \rangle \left\langle \frac{\delta}{\delta u_\beta(x'', t'')} \theta(\underline{x}', t) \, \theta(\underline{x}, t) \right\rangle$$

$$= D^{\alpha\beta}(\underline{x}' - \underline{x}'') \, \delta(t - t'') \left\langle \frac{\delta}{\delta u_\beta(x'', t'')} \theta(\underline{x}', t) \, \theta(\underline{x}, t) \right\rangle$$

$$= D^{\alpha\beta}(\underline{x}' - x'') \left\langle \frac{\delta}{\delta u_\beta(x'', t)} \theta(\underline{x}', t) \, \theta(\underline{x}, t) \right\rangle$$

How to evaluate this functional derivative ?

Let us look at a simple example:

$$\partial_t \phi = u \, g(\phi) + \cancel{\alpha \phi}.$$

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} u(t'') g[\phi(t'')] \, dt'' + \cancel{\alpha \, \phi(t)}$$

$$\frac{\delta \phi(t)}{\delta u(t')} = \int_{t_0}^{t} \frac{\delta u(t'')}{\delta u(t')} g[\phi(t'')] \, dt''$$

$$\left. \begin{aligned} &= g[\phi(t')] && \text{for} \quad t \geq t' \\ &= 0 && \text{for} \quad t < t' \end{aligned} \right\} \text{causality.}$$

$$A = \frac{\delta}{\delta u_\beta(x'', t)} \theta(\underline{x}', t)\,\theta(\underline{x}, t)$$

$$= \theta(\underline{x}, t)\, \frac{\delta\theta(\underline{x}', t)}{\delta u_\beta(\underline{x}'', t)} + \theta(\underline{x}', t)\, \frac{\delta\theta(\underline{x}, t)}{\delta u_\beta(\underline{x}'', t)}$$

$$\partial_t \theta(\underline{x}, t) = -\underline{u}(\underline{x}, t)\cdot\underline{\nabla}\,\theta(\underline{x}, t) + \cdots$$

$$\theta(\underline{x}, t) = -\int \underline{u}(\underline{x}, t')\cdot\underline{\nabla}\,\theta(\underline{x}, t')\,dt'$$

$$\delta\cancel{\theta(x,t)} = -\int u_\alpha(\underline{x}, t')\,\partial_\alpha^{x}\,\theta(\underline{x}, t')\,dt'$$

$$\frac{\delta\theta(\underline{x}, t)}{\delta u_\alpha(\underline{x}'', t)} = -\delta(\underline{x} - \underline{x}'')\,\partial_\alpha^{x}\,\theta(\underline{x}, t')$$

$$A = -\theta(\underline{x}, t)\,\partial_\alpha^{x'}\,\theta(\underline{x}', t) - \theta(\underline{x}', t)\,\partial_\alpha^{x}\,\theta(\underline{x}, t)$$

$$\langle u_\alpha(\underline{x}', t)\,\theta(\underline{x}', t)\,\theta(\underline{x}, t)\rangle$$

$$= \langle\left[-\partial_\beta^{x'}\,\theta(\underline{x})\theta(\underline{x}') - \partial_\beta^{x}\,\theta(\underline{x})\,\theta(\underline{x}')\right] D^{\alpha\beta}(\underline{x} - \underline{x}')\rangle$$

$$= -D^{\alpha\beta}(\underline{x} - \underline{x}')\left(\partial_\beta^{x'} + \partial_\beta^{x}\right)\langle\theta(\underline{x})\,\theta(\underline{x}')\rangle$$

$$\langle(\partial_t\theta')\,\theta\rangle = +D^{\alpha\beta}(\underline{x} - \underline{x}')\left[\partial_\alpha^{x'}\partial_\beta^{x'} + \partial_\alpha^{x'}\partial_\beta^{x}\right]\langle\theta(\underline{x})\,\theta(\underline{x}')\rangle$$

$$\cancel{\partial_t C_2(\underline{x}, t)}$$

$$\partial_t C_2(\underline{x}-\underline{x}', t) = D^{\alpha\beta}(\underline{x}-\underline{x}')\left[\partial_\alpha^{\underline{x}'}\partial_\beta^{\underline{x}'} + \partial_\alpha^{\underline{x}'}\partial_\beta^{\underline{x}}\right] C_2$$

$$+ \varkappa\, \partial_\beta^{\underline{x}'}\partial_\beta^{\underline{x}'} C_2 + \langle f'\theta\rangle$$

$$+ D^{\alpha\beta}(\underline{x}'-\underline{x})\left[\partial_\alpha^{\underline{x}}\partial_\beta^{\underline{x}} + \partial_\alpha^{\underline{x}}\partial_\beta^{\underline{x}'}\right] C_2$$

$$+ \varkappa\, \partial_\beta^{\underline{x}}\partial_\beta^{\underline{x}} C_2 + \langle f\theta'\rangle$$

$$= \Big\{ D^{\alpha\beta}(\underline{x}-x')\left[\partial_\alpha^{\underline{x}}\partial_\beta^{\underline{x}} + \partial_\alpha^{\underline{x}'}\partial_\beta^{\underline{x}} + \partial_\alpha^{\underline{x}}\partial_\beta^{\underline{x}'} + \partial_\alpha^{\underline{x}'}\partial_\beta^{\underline{x}'}\right]$$

$$+ \varkappa\left(\partial_\beta^{\underline{x}}\partial_\beta^{\underline{x}} + \partial_\beta^{\underline{x}'}\partial_\beta^{\underline{x}'}\right)\Big\} C_2 + \langle f'\theta + f\theta'\rangle$$

$$\cancel{\partial_t S_2(r,)}$$

$$\cancel{\partial_t S_2(\underline{x}-\underline{x}', t) = 2\left[\partial_t C_2(0, t) - \partial_t C_2(\underline{x}-\underline{x}', t)\right]}$$

$$\cancel{= 2.\, D^{\alpha\beta}(0)\; 4\, \partial_\alpha^{\underline{x}}\partial_\beta^{x}}$$

$$= \cancel{D^{\alpha\beta}(\underline{x}_i - x_{i'})}\;\partial_\alpha^{\underline{x}_i}$$

$$= \left[\sum_{i,i'=1}^{2} D^{\alpha\beta}(\underline{x}_i - \underline{x}_{i'})\;\partial_\alpha^{\underline{x}_i}\partial_\beta^{\underline{x}_{i'}} + \varkappa\sum_{i=1}^{2}\partial_{\beta\beta}^{\underline{x}_i}\right] C_2$$

$$+ \langle f\theta + f\theta'\rangle$$

$$\boxed{\partial_t C_2 = \mathcal{M}_2 C_2 + \mathcal{F}_2.}$$

In the steady state $\partial_t C_2 = 0$

$$\Rightarrow \qquad \mathcal{M}_2 C_2 = \mathcal{F}_2$$

$$\Rightarrow \qquad C_2 = \mathcal{M}_2^{-1} \mathcal{F}_2$$

~~What about~~ ~~ki~~.

Now let us look at higher order correlation functions.

$$C_{2N}(x_1, \ldots x_{2N}) = \langle \theta(x_1)\, \theta(x_2) \ldots \theta(x_{2N}) \rangle$$

$$\partial_t C_{2N} = \sum_{j=1}^{2N} \langle \theta(\underline{x}_1) \ldots [\partial_t \theta(\underline{x}_j)] \ldots \theta(\underline{x}_{2N}) \rangle$$

$$= \sum \langle \ldots \left[\left\{ - u_\alpha(\underline{x}_j) \partial_\alpha^{x_j} + \varkappa\, \partial_{\beta\beta}^{x_j} \right\} \theta(\underline{x}_j) + f(\underline{x}_j) \right] \ldots \rangle$$

$$= \varkappa \sum_{j=1}^{2N} \partial_{\beta\beta}^{x_j} C_{2N} + \sum_{j,k.} \mathcal{F}(\underline{x}_j - \underline{x}_k)\, C_{2N-2}(x_1, \ldots {}_{j,k} \ldots, x_{2N})$$

$$\cancel{+ \sum^{2N} D^{\alpha\beta}(\underline{x}_j - \underline{y}) \langle \frac{\delta}{\delta u_\beta(\underline{y})} \theta(x_1) \ldots \theta(x_{2N}) \rangle}$$

$$\sum_{j=1}^{2N} D^{\alpha\beta}(\underline{x}_j - \underline{y})\, \partial_\alpha^{x_j} \langle \frac{\delta}{\delta u_\beta(\underline{y})} \theta(\underline{x}_1) \ldots \theta(\underline{x}_{2N}) \rangle$$

$$\cancel{D^{\alpha\beta}(\underline{x}_j - \underline{y}) \partial_\beta^{x_j} \left\langle \frac{\delta}{\delta u_\beta(\underline{y})} \theta(1) \dots \theta(2N) \right\rangle}$$

$$= D^{\alpha\beta}(\underline{x}_j - \underline{x}_k) \partial_\beta^{\underline{x}_j}$$

$$D^{\alpha\beta}(\underline{x}_j - \underline{y}) \partial_\alpha^{\underline{x}_j} \left\langle \frac{\delta}{\delta u_\beta(\underline{y})} \theta(1) \dots \theta(2N) \right\rangle$$

$$= \cancel{D^{\alpha\beta}(\underline{x}_j - \underline{y})}$$

$$= \sum_k D^{\alpha\beta}(\underline{x}_j - \underline{x}_k) \partial_\alpha^{x_j} \partial_\beta^{\underline{x}_k} \langle \theta(1) \dots \theta(2N) \rangle$$

$$\partial_t C_{2N} = \Big[\sum_{j,k} D^{\alpha\beta}(\underline{x}_j - \underline{x}_k) \partial_\alpha^{x_j} \partial_\beta^{\underline{x}_k} + \varkappa \sum_j \partial_{\beta\beta}^{\underline{x}_j} \Big] C_{2N}$$

$$+ \sum_{j,k} \mathcal{F}(\underline{x}_j - \underline{x}_k) C_{2N-2}$$

$$\boxed{\partial_t C_{2N} = \mathcal{M}_{2N} C_{2N} + \Phi_2 C_{2N-2}}$$

~~How can multise~~

$$\partial_t C_{2N} = \mathcal{M}_{2N} C_{2N} + \Phi_2 C_{2N-2}.$$

At steady state; $\mathcal{M}_{2N} C_{2N} = \Phi_2 C_{2N-2}.$

what is the naive scaling dimension of $\mathcal{M}_{2N}$?

$$\mathcal{M}_{2N} = \sum_{j,k} D^{\alpha\beta}(\underline{x}_j - \underline{x}_k)\, \partial_\alpha^{\underline{x}_j} \partial_\beta^{\underline{x}_k} + \kappa \sum_j \partial_{\beta\beta}^{\underline{x}_j}$$

assume zero diffusivity.

Then

$$\mathcal{M}_{2N} \sim \frac{\cancel{r^{-\xi}}}{r^2} \qquad \mathcal{M}_{2N} \sim r^{\xi-2} \mathcal{M}_{2N-2}$$

$$\Rightarrow \quad C_{2N} \sim r^{2-\xi}.\ \Phi_2 C_{2N-2}.$$

$$\Rightarrow \quad \boxed{\begin{aligned} C_2 &\sim r^{2-\xi} \\ C_{2N} &\sim r^{N(2-\xi)} \end{aligned}}$$

which is just ~~di~~ dimensional scaling.

How can this "power-counting" go wrong. ?

(i) The inverse of the operator $\mathcal{M}_{2N}$ is an integral operator. If the integrals do not ~~conserve~~ converge then the different cut-off of the integral might play a role and we might get something other than dimensional contribution

(ii) If there are zero modes

Let us look at convergence properties:

For the second order structure function.

To do this carefully let us first look at $D^{\alpha\beta}(r)$

$$D^{\alpha\beta}(r) = \int \hat{D}^{\alpha\beta}(\underline{k})\, e^{i\underline{k}\cdot\underline{r}}\, d^3k$$

$$= \int \left(\delta_{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}\right) \frac{e^{-(\eta k)^2}\, e^{i\underline{k}\cdot\underline{r}}}{\left(k^2 + k_0^2\right)^{\left(\frac{d+\xi}{2}\right)}}\, d^3k$$

$$= \quad \cancel{D_0}$$

$$= \quad D_0\, \delta^{\alpha\beta} - \frac{1}{2} d^{\alpha\beta}(r)$$

where $D_0 \sim L^{\xi}$.

$$\lim_{\substack{\eta \to 0 \\ k_0 \to 0}} d^{\alpha\beta}(r) = D_1 r^{\xi} \left[(d - 1 + \xi) \delta^{\alpha\beta} - \xi \frac{r^\alpha r^\beta}{r^2} \right]$$

What does this imply for the velocity field?

1. If we ~~st~~ first take the limit of $\eta \to 0$ then the velocity field is no longer ~~differentiable~~ ~~in the~~ expandable in a Taylor series for $\xi < 2$. This is called "rough". velocities are then Hölder continuous with exponent $\frac{\xi}{2} - 0$..

~~A function is called Hölder continuous with exponent~~ α ~~when~~

$$\cancel{\lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}}$$

A function $f(x)$ is called Lipschitz continuous if

$$f(x) - f(y) \leq L |x - y| \quad \text{for } x \to y.$$

where L is a constant.

For non-Lipschitz functions; if a function satifies

$$f(x) - f(y) \leq C |x - y|^\alpha \quad \text{as } x \to y.$$

Then the function is called "α-Hölder"

Thus ~~is the~~ when we are looking for multiscaling of the Kraichnan model we are looking at a system where the advection is by velocities which are ~~not Hö~~ only $\frac{\xi}{2}$ Hölder. Except when $\xi = 2$; when the ~~behavi~~ velocity is smooth.

With this form for $D^{\alpha\beta}(r)$; we have

$$\cancel{M_2(r) = \sum_{1,2} D^{\alpha\beta}(r)\, \partial_\alpha^{x_1} \partial_\beta^{x_2} + \kappa}$$

$$M_2(r) = \sum_{\substack{i=1,2 \\ j=1,2}} D^{\alpha\beta}(r)\, \partial_\alpha^{x_i} \partial_\beta^{x_j} + \kappa \sum_{i=1,2} \partial_{\beta\beta}^{x_i}$$

~~For $i \neq j$~~

$$\cancel{M_2(r) = D_0 \nabla^2 + \text{other terms}}.$$

If you now take the limit of $L \to \infty$, or $k_0 \to 0$ ~~so~~ the $i=j$ part of $M_2(r)$ blows up. The ~~one~~ way to tackle this problem is to use the structure function instead of the correlation function.

~~The equa~~

In steady state C_2 satisfies the equation:

$$-M_2 C_2 = \mathcal{F}_2$$

$$\Rightarrow \left[\sum_{i,j=1}^{2} D^{\alpha\beta}(\underline{x}_i - \underline{x}_j)\, \partial_\alpha^{\underline{x}_i} \partial_\beta^{\underline{x}_j} + \varkappa \sum_{i=1}^{2} \partial_{\beta\beta}^{\underline{x}_i} \right] C_2(\underline{x}_i, \underline{x}_j) = \mathcal{F}_2(\underline{x}_i, \underline{x}_j)$$

- Translational invariance $C_2(\underline{x}_i, \underline{x}_j) = C_2(\underline{x}_i - \underline{x}_j)$

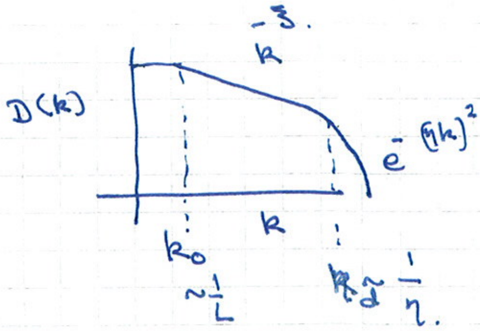
- Isotropy $C_2(\underline{x}_i - \underline{x}_j) = C_2(|\underline{x}_i - \underline{x}_j|) = C_2(r)$

$$-r^{1-d} \partial_r \left[(d-1) D_1 r^{d-1+\xi} + 2\varkappa r^{d-1} \right] \partial_r C_2(r) = \mathcal{F}_2\left(\frac{r}{L}\right)$$

$$\Rightarrow \quad C_2(r) = \frac{1}{(d-1) D_1} \int_r^\infty \frac{x^{1-d}\, dx}{x^\xi + \eta^\xi} \int_0^x \mathcal{F}_2\left(\frac{y}{L}\right) y^{d-1}\, dy.$$

with the boundary condition ~~$C_2(0)$ is finite~~
$C_2(\infty)$ is zero.

Let us look at the consequences:



- For fixed L, and fixed η

with $\eta \ll r \ll L$

$$C_2(r) \approx \frac{\Phi(0)\, L^{2-\xi}}{(2-\xi)(d+\xi-2)(d-1)D_1} - \frac{\Phi(0)}{(2-\xi)\, d(d-1)D_1}$$

$$\Rightarrow \qquad S_2(r) \approx \frac{2}{(2-\xi)\, d\, (d-1)D_1}\, \Phi(0)\, r^{2-\xi}.$$

$$\boxed{\zeta_2 = 2-\xi.}$$

- For $r \ll \eta$

$$S_2(r) \approx \frac{1}{2\varkappa d}\, \Phi(0)\, r^2$$

Then the only way multiscaling can appear is by zero modes.

~~who~~

$$\mathcal{M}_{2N} C_{2N} = - \Phi_2 C_{2N-2}.$$

Only the "zero modes" of $\mathcal{M}_{2N}$ can escape dimensional prediction. But to see multiscaling the following should also be true.

1. The "zero modes" should have scaling properties. (e.g. constant are not useful)

2. The "zero modes" should satisfy the right boundary condition.

3. Each order should have its own zero modes which cannot be constructed out of the zero modes of smaller orders.

~~4. It turns out that~~

~~4. It turns out that~~

4. The scaling exponents of the zero modes should be less than the dimensional exponent.

Zero modes and multiscaling

$$\xi_N = N$$

$$\cancel{S_{2N} = N(2-\xi) - \frac{2N(2N-2)}{2(d+2)}\xi + O(\xi^2)}$$

There are three well behaved limits

$$\xi = 0 \,, \qquad \xi = 2$$

and $d \to \infty$

There exists three perturbation expansions from these three directions

$$\zeta_{2N} = N(2-\xi) - \frac{2N(N-1)}{(d+2)}\xi + O(\xi^2)$$

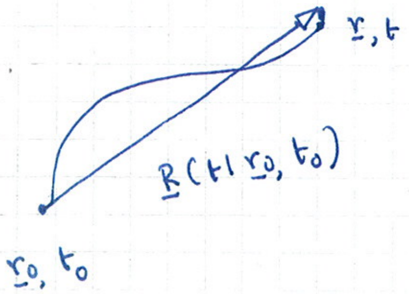
(Bernard, Gawedzki, Kupiainen 1996)

$$\zeta_{2N} = N(2-\xi) - \frac{2N(N-1)}{d}\xi + O\left(\frac{1}{d^2}\right)$$

(Chertkov & Falkovich 1996)

~~The thr.~~

Particle and fields.



$$\underline{v}(t \mid \underline{r}_0, t_0) = \frac{d\underline{R}}{dt} \quad - \text{ Lagrangian velocity.}$$

$$\underline{u}(\underline{r}, t) = \underline{v}(t \mid \underline{r}_0, t_0) \quad \text{when} \quad \underline{r}_0 + \underline{R} = \underline{r}$$

$$= \frac{d\underline{R}}{dt} \qquad - (1)$$

~~Now we prove~~

$\Rightarrow$ $\cancel{\boxed{d\underline{R} = \underline{u}(\underline{r}, t)\, dt}}$

~ ~~Stochastic ODE~~.

~~If $\underline{R}$~~

<u>Statement</u>. If $\underline{R}(t \mid t_0, \underline{r}_0)$ solves eqn. (1) with the initial condition that $\underline{R}(t_0 \mid t_0, r_0) = \underline{r}_0$ & then

$$\theta(t, \underline{r}) = \int \delta\left[\underline{r} - \underline{R}(t \mid t_0, \underline{r}_0)\right] \theta(t_0, \underline{r}_0)\, d\underline{r}_0$$

solves the equation

$$\partial_t \theta + (\underline{u} \cdot \underline{\nabla})\theta = 0.$$

~~Proof. let us first to prove this in one one dimension.~~

$$\cancel{u(x,t)} \quad \cancel{\frac{da}{dt} = u(x,t)}$$

~~Let $a(t\,|\,x_0, t_0)$ be a solution of the preceeding equation~~

~~and $\theta($~~

Meaning of the above statement is to be understood in the "weak" sense; or in the sense of "distributions."

~~i.e.~~ i.e. for ~~$\psi(x)$~~ ~~in~~ a ~~class~~ ~~of~~ ~~to~~ $\psi(\underline{r})$ in

= ~~An aside~~ ~~on~~ ~~[illegible]~~

a class of test functions

$$\int \psi(\underline{r})\, \theta(t, \underline{r})\, d\underline{r} = \int \delta[\underline{r} - \underline{R}]\, \psi(\underline{r})\, \theta(t_0, \underline{r_0})\, d\underline{r_0}\, d\underline{r}$$

and this solves the equation

$$\int \psi(\underline{r})\, \partial_t \theta \, d\underline{r} + \int \psi(\underline{r})\, (\underline{u} \cdot \underline{\nabla})\, \theta \, d\underline{r} = 0 .$$

~~Exa~~ " weak solution "

A careful way of saying that we are allowing ~~st~~ " non-functions" like δ functions ~~to~~ as solutions to the differential equations. Let us look at more familiar example.

Poisson eqn ~~for a point~~ ~~mass~~ charge ~~is~~

$$\nabla^2 \phi(\underline{r}) = -\frac{\rho(\underline{r})}{\epsilon_0}$$

For a point ~~mass~~. charge we know what the solution is; it is

$$\cancel{\phi(r) = -\frac{q}{r}, \quad \rho(\underline{r}) = q\,\delta(\underline{r})}$$

$$\phi(\underline{r}) = +\frac{q}{4\pi\epsilon_0}\frac{1}{r}, \quad \rho(\underline{r}) = q\,\delta(\underline{r})$$

$$+\frac{\cancel{q}}{4\pi\cancel{\epsilon_0}}\nabla^2\left(\frac{1}{r}\right) = -\frac{\cancel{q}}{\cancel{\epsilon_0}}\delta(\underline{r})$$

or

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta(\underline{r})$$

This is to be interpreted in the " weak" sense. i.e.

$$\int \psi(\underline{r})\,\nabla^2\phi(\underline{r})\,d\underline{r} = -\int \frac{\rho(\underline{r})}{\epsilon_0}\,\psi(\underline{r})\,d\underline{r}$$

<u>Proof</u> $\quad \frac{da}{dt} = u(x, t)$ $\quad$ <u>in 1-d</u>

and $\quad \theta(x, t) = \int \delta[x - a(t \mid y, t_0)] \cancel{dy}\, \theta(y, t_0)\, dy$

$$\frac{\partial}{\partial t} \int \psi(x)\, \theta(x, t)\, dx = \iint \psi(x) \frac{\partial}{\partial t} \delta[x - a(t \mid y, t_0)]\, \theta(y, t_0)\, dy\, dx$$

$$= -\iint \psi(x)\, \delta'[x - a] \frac{da}{dt}\, \theta(y, t_0)\, dy\, dx$$

$$= -\iint u(x)\, \psi(x)\, \delta'[x - a(t \mid y, t_0)] \cancel{dy\,d}\, \theta(y, t_0)\, dy\, dx$$

$$= \int \frac{d}{dx}\left[u(x)\, \psi(x)\right] \int \delta[x - a(t \mid y, t_0)]\, \theta(y, t_0)\, dy\, dx$$

$$= \int \frac{d}{dx}\left[u(x)\, \psi(x)\right] \theta(x, t)\, dx$$

$$= -\int \psi(x)\, u(x) \frac{d}{dx} \theta(x, t)\, dx$$

$$\Rightarrow \quad \frac{\partial \theta}{\partial t} = - u(x) \frac{d\theta}{dx}$$

in several dimensions $\quad \frac{\partial \theta}{\partial t} = -(\underline{u} \cdot \nabla)\, \theta$

Statement

If we now consider $\underline{R}(t \mid \underline{r}_0, t_0)$ to be a solution of the equation

$$\frac{d\underline{R}}{dt} = \underline{u}(\underline{x}, t) + \sqrt{2\varkappa}\, \underline{\eta}.$$

where η is white noise. then

$$\theta(t, \underline{r}) = \Big\langle \int \delta\left[\underline{r} - \underline{R}(t \mid \underline{r}_0, t_0)\right] \theta(t_0, r_0)\, d\underline{r}_0 \Big\rangle$$

over η.

satisfies the following PDE

$$\partial_t \theta + (\underline{u} \cdot \underline{\nabla})\theta = \varkappa \nabla^2 \theta.$$

Before we proceed for a proof we have to understand the notion of SDE.

$$d\underline{R} = \underline{u}(\underline{r}, t)\, dt + \sqrt{2\varkappa}\, d\underline{\beta}$$

where $$d\underline{\beta} = \int_t^{t+\delta t} \underline{\eta}\, dt$$

$$W^\alpha_\beta(t \mid t_0, \underline{r}_0) = \frac{\partial}{\partial r_0{}^\beta} R^\alpha(t \mid t_0, \underline{r}_0)$$

$$\underline{B}(t, \underline{r}) = \int \delta[\underline{r} - \underline{R}]\, \underline{B}(t_0, r_0)\, d\underline{r}_0$$

$$\partial_t \underline{B} + (\underline{u} \cdot \underline{\nabla})\underline{B} + \underline{B}(\underline{\nabla} \cdot \underline{u}) - (\underline{B} \cdot \underline{\nabla})\underline{u} - \varkappa \nabla^2 \underline{B} = 0$$

Stochastic Differential Equations

So first consider the SDE in one dimension

$$da = u\,dt + \sqrt{2\varkappa}\,d\beta .$$

consider

$$d[f(a)]_{\text{Ito}} = f[a(t+\delta t)] - f[a(t)]$$

$$= f[a+da] - f[a]$$

$$= \cancel{f(a)} + \frac{df}{da}.da + \frac{1}{2}\frac{d^2f}{da^2}(da)^2 + O(da^3)$$

then

$$- \cancel{f(a)}$$

$$= \frac{df}{da}da + \frac{1}{2}\frac{d^2f}{da^2}.(da)^2 + O(da^3)$$

$$da^2 = u^2 dt^2 + 2u\,dt\,d\beta\sqrt{2\varkappa} + 2\varkappa\, d\beta^2$$

$$d\beta^2 = \int_t^{t+dt}\int_t^{t+dt} \eta(t_1)\,dt_1 \;\; \eta(t_2)\,dt_2$$

$$\langle d\beta^2 \rangle = \iint \langle \eta(t_1)\,\eta(t_2)\rangle\, dt_1\, dt_2 = dt$$

$$[df(a)]_{\hat{\text{I}}\text{to}} = \frac{df}{da}(u\,dt + \sqrt{2\varkappa}\,d\beta) + \varkappa\frac{d^2f}{da^2}\,dt$$

$$[df(a)]_{\text{Stratanovich}} = \frac{df}{da}\left(u\,dt + \sqrt{2\varkappa}\,d\beta\right)$$

proof

$$da = \sqrt{2\varkappa}\, d\beta .$$

$$\theta(x,t) = \int \delta\left[x - a(t \mid y, t_0)\right] \cancel{dy}\, \theta(y, t_0)\, dy$$

or

$$\int \psi(\cancel{x,t})\, \theta(x,t)\, dx = \int \psi(x)\, \delta\left[x - a(t)\right] \theta(y, t_0)\, dy\, dx$$

$$\int \psi \frac{\partial \theta}{\partial t}\, dx = \cancel{\int \psi(x)\, \delta'\left[x-a\right] \frac{da}{dt}\, \theta(y,t_0)\, dy}$$

$$= \frac{\partial}{\partial t} \int \psi(a)\, \theta(y)\, dy$$

$$\cancel{\varkappa} \quad \cancel{\delta'}$$

$$= \cancel{\frac{d\psi}{da}}$$

$$= \int \varkappa \frac{d^2\psi}{da^2} \cdot \theta(y)\, dy$$

$$= \int \varkappa\, \theta(y) \frac{d}{da^2}\, \psi \quad dy$$

$$= \int \varkappa\, \theta(y) \frac{d^2}{da^2} \cdot \int \psi(x)\, \delta(x-a)\, dx \quad dy$$

$$= \iint \varkappa \quad \theta(y)\, \psi(x) \frac{d^2}{da^2}\, \delta(x-a) \quad dx\, dy$$

$$= \iint \varkappa \quad \psi(x) \frac{d^2}{da^2}\, \delta(x-a)\, \theta(y) \quad dx \quad dy$$

$$= \iint \varkappa\, \psi(x) \frac{d^2}{dx^2} \cdot \delta(x-a)\, \theta(y)\, dx \quad dy$$

$$= \int \varkappa\, \psi(x) \frac{d^2}{dx^2} \cdot \theta(x) \quad dx \qquad \text{Q. E. D.}$$

Particle picture

$$d\underline{R} = \underline{u}(\underline{R}, t)\, dt + \sqrt{2\varkappa}\, d\beta(t)$$

For $\underline{u} = 0$ this is just the Langvein eqn.

$$\Delta\underline{R}(t) = \underline{R}(t) - \underline{R}(0)$$

The probability of having a value ΔR, $P[\Delta R]$ satisfies the eqn

$$[\partial_t - \varkappa \nabla^2] P = 0.$$

$$P(\Delta\underline{R}; t) = \frac{1}{(4\pi\varkappa t)^{-d/2}} \exp\left[-\frac{(\Delta\underline{R})^2}{4\varkappa t}\right]$$

For $\varkappa = 0$.

$$\frac{d}{dt}\langle(\Delta\underline{R})^2\rangle = 2\int_0^t \langle \underline{V}(0)\cdot\underline{V}(s)\rangle\, ds$$

proof

$$\frac{da}{dt} = V(t \mid y, t_0)$$

$$a(t) = \int_0^t V(s)\, ds$$

$$[a(t) - a(0)]^2 = \iint V(t_1)V(t_2)\, dt_1\, dt_2$$

$$+ \cancel{2\, V(t)\, V(0)}$$

$$+ 2.\, a(0)\int_0^t V(t_1)\, dt_1$$

$$+ a^2(0)$$

$$\Rightarrow \frac{d}{dt}\langle[a(t) - a(0)]^2\rangle = 2\int_0^t \langle V(t)V(t_2)\rangle dt_2$$

$$= 2\int_0^t \langle V(0)V(s)\rangle\, ds$$

$$\tau = \frac{\int_0^\infty \langle \underline{v}(0) \cdot \underline{v}(s) \rangle \, ds}{\langle v^2(0) \rangle}$$

- Divergence of τ implies persistence of correlations.

- If the ~~above~~ above integral converges

$$\frac{d}{dt}\langle (\Delta R)^2 \rangle = 2\langle v^2 \rangle \tau \Rightarrow \boxed{\langle (\Delta R)^2 \rangle = 2\langle v^2 \rangle \tau t}$$

— Diffusion.

- For small time $t < \tau$.

$$\frac{d}{dt}\langle (\Delta R)^2 \rangle = 2\langle v^2 \rangle t \Rightarrow \boxed{(\Delta R)^2 = \cancel{2}\langle v^2 \rangle \frac{t^2}{\cancel{2}}}$$

— Ballistic.

- For $t \gg \tau$, and for finite τ; (ΔR) behaves like sum over many random variables. Then (ΔR) behaves like Brownian motion in ~~three~~ d dimensions.

- Divergence of the integral implies super diffusion.

Two particle separation:

$$\frac{da}{dt} = \sigma a$$
$$\int \frac{da}{a} = \int \sigma\, dt$$
$$\ln a = \sigma t$$
$$a =$$

$$\underline{R}_{12} = \underline{R}_1 - \underline{R}_2$$

- smooth velocities. (Batchelor regime)

$$\frac{d\underline{R}_{12}}{dt} = \underline{u}(\underline{R}_1) - \underline{u}(\underline{R}_2)$$

$$\Rightarrow \quad \frac{d}{dt} R^{\alpha}_{12} = \sigma^{\alpha\beta} R^{\beta}_{12} \quad \text{with} \quad \sigma^{\alpha\beta} = \frac{\partial u^{\alpha}}{\partial x^{\beta}}$$

$\sigma^{\alpha\beta} \rightarrow$ strain matrix.

$$\Rightarrow \quad \boxed{R^{\alpha}_{12}(t) = W^{\alpha\beta}(t)\, R^{\beta}(0)}$$

First in 1-dimension:

$$\delta_{12} = Wt$$

$$\dot{a}_{12} = \sigma a_{12}$$

$$\Rightarrow \quad \ln\left[\frac{a_{12}(t)}{a_{12}(0)}\right] = \ln\left[W(t)\right] = \int_0^t \sigma(s)\,ds \equiv X. \qquad X = \sum_{i=1}^{N} Y_i\,, \quad N = \frac{t}{\tau}.$$

For $t \gg \tau$ the correlation time of σ, X is a sum of many IID random numbers. $P(x)$ has a large-deviation form.

$\langle X \rangle = N \langle y \rangle$ grows linearly with time.
Or the interparticle distance grows exponentially with time. with exponent

$$\lambda = \frac{\langle x \rangle}{t}. \quad \text{- Lypunoff exponent.}$$

In several dimension

$$W(t) = T \exp\left[\int_0^t \sigma(s)\, ds\right]$$

$$= \sum_{n=0}^{\infty} \int_0^t \sigma(s_n)\, ds_n \ldots \int_0^{s_3} \sigma(s_2)\, ds_2 \int_0^{s_2} \sigma(s_1)\, ds_1$$

$$\lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln |W f_i| \quad \text{— Lypunoff exponents.}$$

In compressibility $\Rightarrow$ $\det(W) = 1$, $\sum \lambda_i = 0$.

implies volume is preserved.

Consider $I = W^T W$.

For delta correlated strain

$$\cancel{\sigma^{\alpha\beta}\, \sigma^{\gamma\delta}} \quad \langle \sigma^{\alpha\beta} \sigma^{\mu\nu} \rangle = 2\, \delta(t-t')\, C^{\alpha\beta\mu\nu}$$

$$\left\langle \frac{\partial u^\alpha}{\partial x^\beta} \cdot \frac{\partial u^\mu}{\partial x^\nu} \right\rangle = \frac{\partial}{\partial x^\beta} \frac{\partial}{\partial x^\nu} \cdot \langle u^\alpha u^\mu \rangle = \partial_\beta \partial_\nu D^{\alpha\mu}\, \delta(t-t')$$