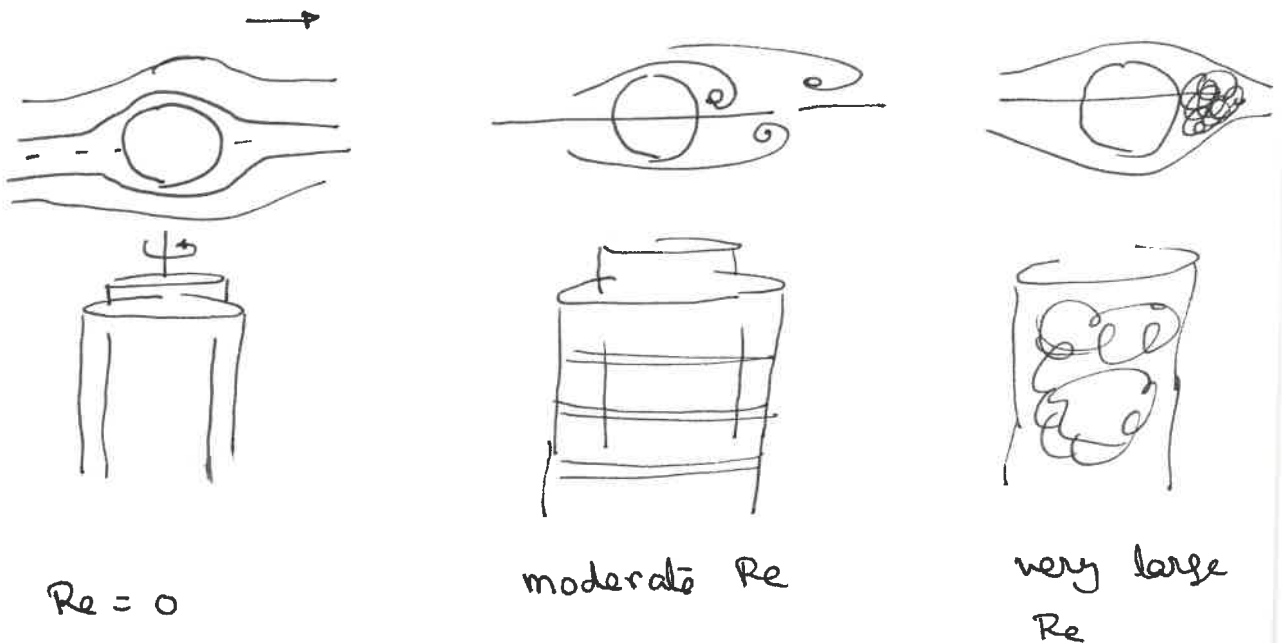


①

Turbulence :

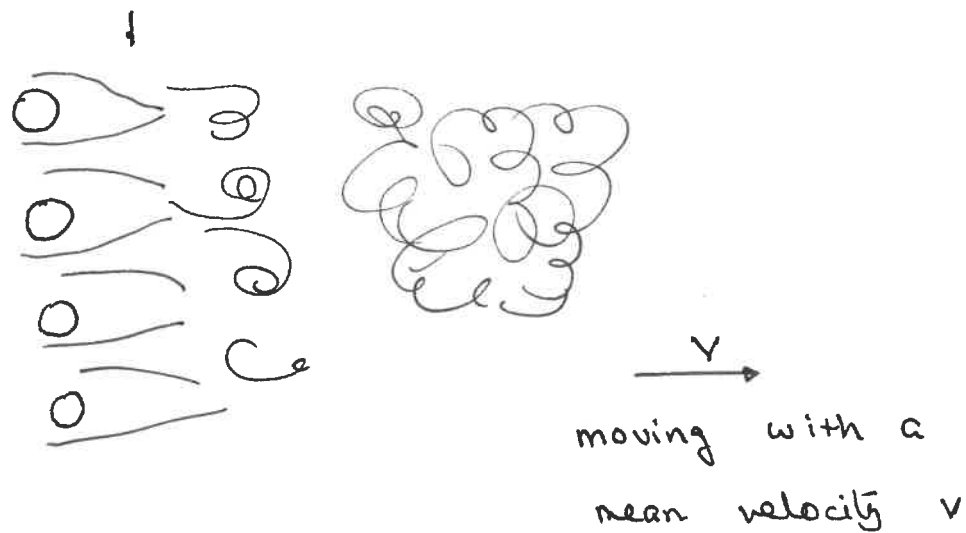
x Need for a statistical approach.

- As Reynolds number increases flow becomes a function of space and time that fluctuates ~~very fast in time~~ and ~~on~~ in both space and time?
- As Re increases symmetries are broken i.e. patterns emerge. These patterns themselves become unstable till all symmetries are statistically restored.



(2).

Homogeneous and isotropic turbulence: (HIT)



Philosophical ~~quers~~ question:

Q. Navier-Stokes is a deterministic equation.

How can that give rise to statistical (random) solution?

A. The answer is not obvious or easy. Partial answer comes from studying non-linear dynamical systems. Simply put: because of non-linearities. But ~~not~~ not all non-linear equations ~~satisfy~~ give "random" solutions.

③.

= Statistical description of flows:

$\vec{u}(\vec{x}, t)$ is a random function.

$\Rightarrow$ It has a probability distribution
function: $P[\vec{u}(\vec{x})]$

$\uparrow$
non-trivial

Probability for a function (not a variable)

and correlation functions:

$$C_1^\alpha = \langle u_\alpha(\vec{x}) \rangle$$

$$C_2^{\alpha\beta} = \langle u_\alpha(\vec{x}) u_\beta(\vec{x} + \vec{r}) \rangle$$

$$C_3^{\alpha\beta\gamma} = \langle u_\alpha(\vec{x}) u_\beta(\vec{x} + \vec{r}_1) u_\gamma(\vec{x} + \vec{r}_2) \rangle$$

$\vdots$

gets more and more complicated.

$\vdots$

How many do I need?

④.

Statistically



Use symmetries:

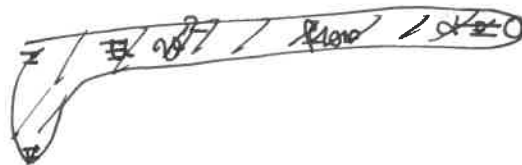
Homogeneous & Isotropic

$$C_1 = \langle \vec{u}(\vec{x}) \rangle : \text{constant.}$$

$$C_2^{\alpha\beta}(\vec{r}) = \langle u_\alpha(\vec{x}) u_\beta(\vec{x} + \vec{r}) \rangle$$

$$C_2^{\alpha\beta}(\vec{r} = 0) = \langle u_\alpha(\vec{x}) u_\beta(\vec{x}) \rangle$$

$$= 0 \quad \text{for } \alpha \neq \beta \quad \text{Known experimentally.}$$

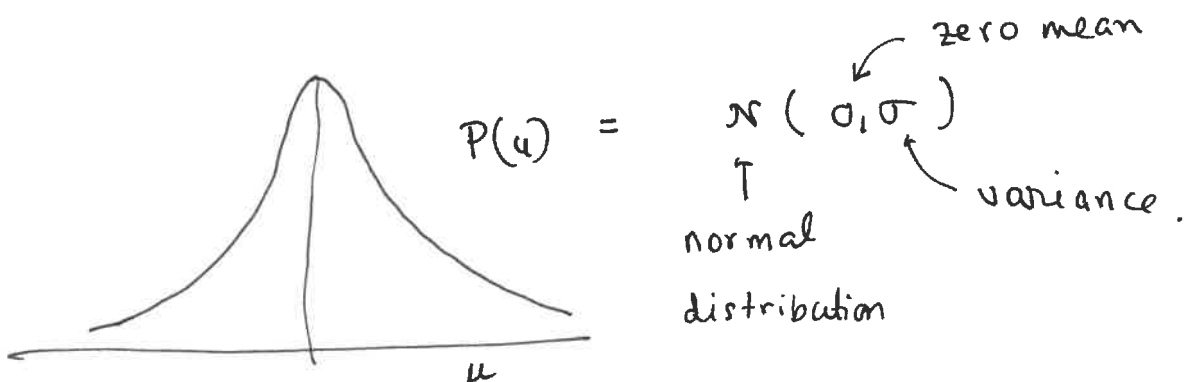


Consider: $P(u)$ at a fixed point $\vec{x}$.

(a) cannot depend on $\vec{x}$ (by symmetry)

(b) Is known to be a Gaussian (experimentally)

$$P(u) = \exp\left(-\frac{u^2}{\sigma^2}\right)$$



⑤.

Averaging : In the pr.

Some stationary state ; energy is constant; the fluid is being stirred but the rate of energy injection is equal to the rate of energy dissipation by viscosity ; one average. §

$$\partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} = \nu \nabla^2 \vec{u} - \nabla p + \vec{f} \quad \text{stirring}$$

$$\nabla \cdot \vec{u} = 0$$

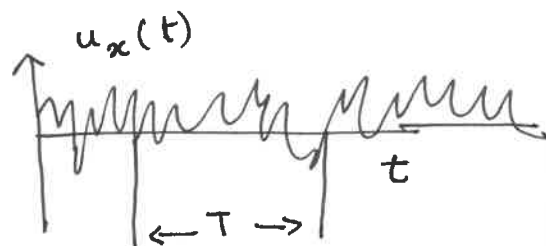
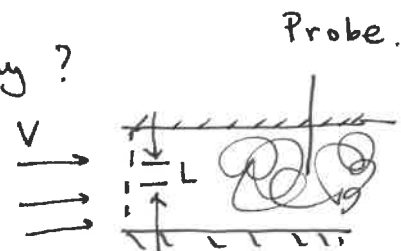
Now consider :

$$C_2^{\alpha\beta}(\vec{r}) = \langle u^\alpha(\vec{x}) u^\beta(\vec{x} + \vec{r}) \rangle$$

what does the experiments say ?

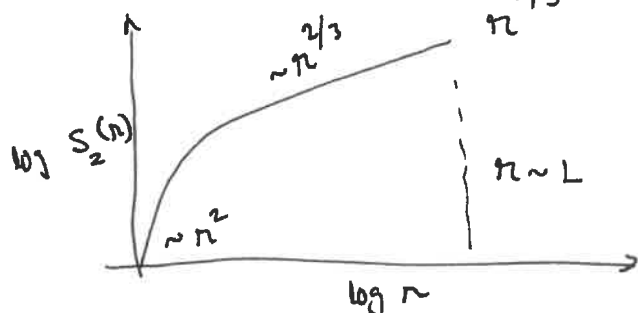
Also

$$S_2(r) = \langle [u(t + \frac{r}{V}) - u(t)]^2 \rangle$$



$$r = VT$$

$\sim r^2$ for small r
 $\sim r^{2/3}$ for intermediate r
 $r \sim L$



Implies :

$$\begin{aligned}
 \underline{1.} \quad & \langle [u(x+n) - u(x)]^2 \rangle \\
 = & \langle u^2(x+n) \rangle + \langle [u(x)]^2 \rangle - 2 \langle u(x+n)u(x) \rangle \\
 & \quad \quad \quad \uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow \\
 & \quad \quad \quad \text{does not} \quad \text{depend} \quad \text{depends on } n \\
 & \quad \quad \quad \text{on } n \\
 & \quad \quad \quad (\text{homogeneity})
 \end{aligned}$$

$$= [u^2(x)] - 2 \langle u(x+n)u(x) \rangle$$

$\Rightarrow S_2(n)$ and $C_2(n)$ has the same n dependence.

2. Why consider $S_2(n)$? It is Galilean invariant but $C_2(n)$ is not.

3. Assume $u(x+n)$ is Taylor expandable

$$\Rightarrow u(x+n) = u(x) + \frac{\partial u}{\partial x} n + O(n^2)$$

$$\Rightarrow S_2(n) = \left\langle \left[\frac{\partial u}{\partial x} n \right]^2 \right\rangle + O(n^4)$$

$\uparrow$ does not depend on n

$$= () n^2$$

explains the small n behaviour.

4 what about $\pi^{2/3}$?

This is tricky. Explained by Kolmogorov's theory.
Proceed cautiously.

$$\partial_t \vec{u} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = \nu \nabla^2 \vec{u} - \vec{\nabla} p + \vec{f}$$

multiply by $\vec{u}$ and integrate over all space.

$$E = \left\langle \frac{u^2}{2} \right\rangle$$

$$\partial_t E = -\nu \underbrace{\langle \omega^2 \rangle}_{\text{dissipation}} + \underbrace{\langle \vec{f} \cdot \vec{u} \rangle}_{\text{energy injection}}$$

$$\vec{\omega} = \vec{\nabla} \times \vec{u}$$

(This is part of homework)

In stationary state : $\partial_t E = 0$

$$\nu \langle \omega^2 \rangle = \langle \vec{f} \cdot \vec{u} \rangle \equiv \varepsilon \quad \text{energy dissipation rate.}$$

Hypothesis 1 : zero-th law of turbulence.

As $Re \rightarrow \infty$, as $\nu \rightarrow 0$ (hold $\vec{f}$ constant)

$$\varepsilon = \lim_{\nu \rightarrow 0} \nu \langle \omega^2 \rangle$$

remains non-zero
but reaches a finite
limit

4.9 ε remains finite as $\nu \rightarrow 0$.

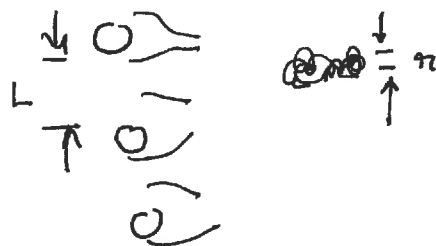
A remarkable result. As $\nu \rightarrow 0$ we do not get the Euler equation. $\nu \rightarrow 0$ limit and $\nu = 0$ are not the same. The $\nu \rightarrow 0$ limit is singular.

Think about $S_2(\pi)$. What can it depend on?

4.b It may depend on ν ? No, not in the limit $\nu \rightarrow 0$

4.c It may depend on $\vec{f}$? Maybe not. Typically $\vec{f}$ is limited to the large scales.

4.d Then the only quantities $S_2(\pi)$ can depend on are: π and ε



(9)

$$S_2(n) = f(n, \varepsilon)$$

$$\Rightarrow S_2(n) = C(\varepsilon n)^{2/3}$$

↑
some
dimensionless
constant.

More generally

$$S_p(n) = \langle [u(x+n) - u(x)]^p \rangle$$

$$\sim (\varepsilon n)^{p/3}$$

$$\varepsilon = \langle \vec{f} \cdot \vec{u} \rangle$$

$$\partial_t \vec{u} = \dots + \vec{f}$$

$$\Rightarrow [f] = \frac{U}{T} = \frac{L}{T^2}$$

$$\Rightarrow [\varepsilon] = \frac{L^2}{T^3}$$

$$[S_2(n)] = \frac{L^2}{T^2}$$

$$[\varepsilon n] = \frac{L^3}{T^3}$$

What are the consequences?

- Eddies: Turbulence consists of vortices ($\vec{\nabla} \cdot \vec{u} = 0$)
velocity fluctuations of length scale 'n' are
called eddies of size 'n'.

$$\delta u(n) \sim (\varepsilon n)^{1/3}$$

- Lifetime of eddies:

Lagrangian



$$\tau_n \sim \left(\frac{n^2}{\varepsilon} \right)^{1/3}$$

$$\Rightarrow \boxed{\tau_n \sim n^{2/3}}$$

Larger eddies live longer.



Eulerian

$$u(t+T) = u(t + \frac{n}{v})$$

$$\Rightarrow T \sim n$$

• Energy spectrum: $\tilde{u}_\alpha(\vec{k}) = \int u_\alpha(\vec{x}) e^{i\vec{k} \cdot \vec{x}} d^d x$

(10)

~~E(k)~~

$$S_2(n) = \langle u_\alpha(\vec{x}) u_\alpha(\vec{x} + \vec{n}) \rangle = \frac{1}{2} S_2(\vec{n}) \stackrel{\uparrow}{=} S_2(n) \text{ by isotropy}$$

~~E(k)~~ $\approx S_2(kn)$

~~E(k)~~ $\propto \tilde{u}(\vec{k}) \tilde{u}^*(\vec{k})$

$$E(\vec{k}) = \langle \tilde{u}_\alpha(\vec{k}) \tilde{u}_\alpha(\vec{k})^* \rangle$$

$$= \int \langle u_\alpha(\vec{x}) u_\alpha^*(\vec{y}) \rangle e^{i\vec{k} \cdot \vec{x} - i\vec{k} \cdot \vec{y}} d^d x d^d y$$

$$= \int S_2(\vec{n}) e^{i\vec{k} \cdot \vec{n}} d^d n d^d(x+y)$$

$$= \int S_2(n) \exp(i k n \sin \theta) d\Omega n^2 dr d^d(x+y)$$

$$\begin{aligned} & \propto \frac{S_2}{k} \\ & \sim \int \frac{\xi^{2/3}}{k^{2/3}} \exp(i \xi \sin \theta) d\Omega \frac{\xi^2}{k^2} \frac{d\xi}{k} d^d(x+y) \left| \begin{array}{l} kn \sim \xi \\ dr \sim \frac{d\xi}{k} \\ n^{2/3} \sim \frac{\xi^{2/3}}{k^{2/3}} \end{array} \right. \\ & \sim k^{-11/3} \end{aligned}$$

$$E(k) = E(\vec{k}) 4\pi k^2 \sim k^{-5/3}$$

• Dissi

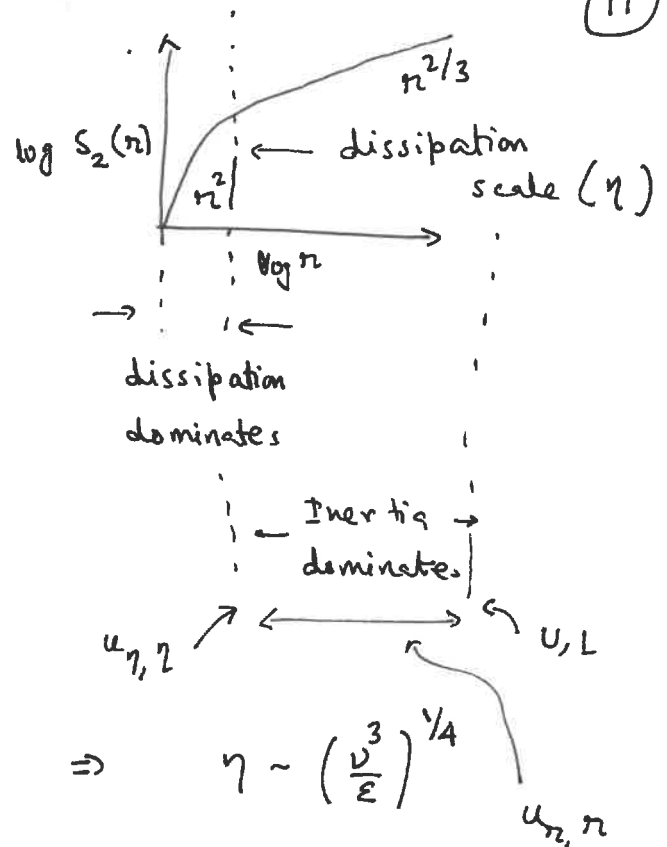
• Dissipation scale:

$$\Rightarrow \frac{\delta u^2}{\eta} \sim \nu \frac{\delta u}{\eta^2}$$

$$\Rightarrow \eta \sim \frac{\nu}{\delta u(\eta)}$$

$$\Rightarrow \eta \sim \frac{\nu}{\varepsilon^{1/3} \eta^{1/3}}$$

$$\Rightarrow \eta^{4/3} \sim \nu \varepsilon^{-1/3}$$



$$\Rightarrow \eta \sim \left(\frac{\nu^3}{\varepsilon} \right)^{1/4}$$



$$u_\eta \sim (\varepsilon \eta)^{1/3}; \quad Re_\eta \sim \frac{\eta u_\eta}{\nu} \sim \frac{\nu^{3/4} \varepsilon^{1/3}}{\varepsilon^{1/4}} \frac{\varepsilon^{1/4}}{\varepsilon^{1/12}} \frac{1}{\eta} \sim 1$$

$$Re_n \sim \frac{u_n n}{\nu} \sim \frac{\varepsilon^{1/3} n^{2/3}}{\nu} \sim n^{2/3}$$

$$Re_L = \frac{UL}{\nu} - \text{Large}$$

As we go to smaller and smaller scales the scale dependent Re_n decreases and becomes order unity at the dissipation scales.

• Scale-dependent ~~viscosity~~ viscosity:

$$\frac{u_n^2}{n} \sim \nu_n \frac{u_n}{n^2} \Rightarrow$$

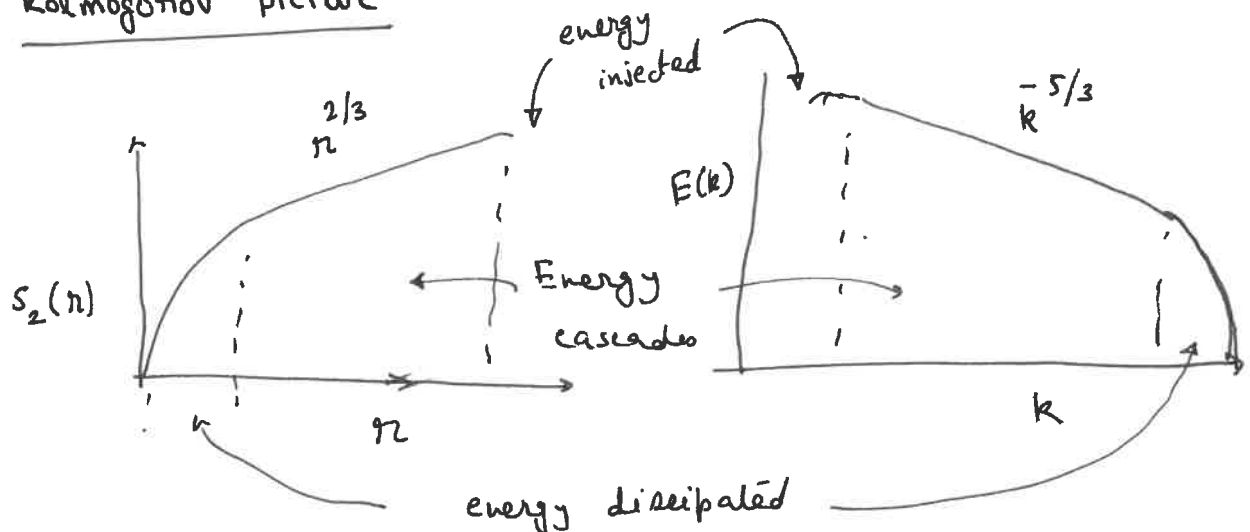
$$\boxed{\nu_n \sim n u_n \sim \varepsilon^{1/3} n^{2/3}}$$



Note: $\frac{u_L^3}{L} \sim \varepsilon$

$$\nu_L \sim UL - \text{turbulent viscosity not small.}$$

Kolmogorov picture



Cascade $\Leftrightarrow$ Inertia dominates $\Leftrightarrow$ any physical quantity is determined by ϵ .

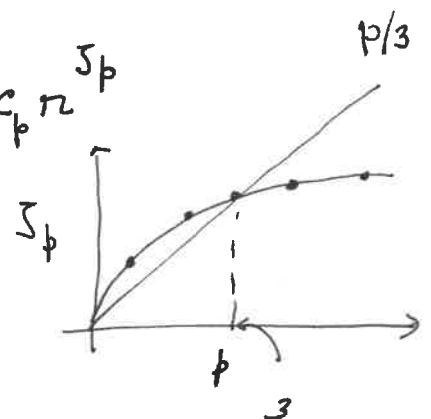
Caveats :

1. $S_2(n) = C (\epsilon n)^{2/3}$

↑
Universal in Kolmogorov scenario but not in ~~real~~ reality.

2. In ~~real~~ reality : $S_p(n) \approx C_p n^{\zeta_p}$

Kolmogorov implies : $\zeta_p = p/3$



~~This~~ This is THE unsolved problem of turbulence.

We believe ζ_p are universal but C_p are not.

Note: Universal: Does not depend on how the fluid is stirred, and what the viscosity is.

Properties of Inertial range:

1. Universal.
2. scale invariant (power-law scaling)

How to generalize:

1. Look for conserved quantities. (energy)
2. Scale-separation and cascade of conserved quantity.
(Energy injected at large scale; dissipated at small scales)
3. Dimensional argument involving the flux of the conserved quantity.

Astrophysical implications:

1. Almost all astrophysical flows are turbulent.
2. Applies to small (inertial) scales where isotropy and homogeneity can be assumed.
3. Only for incompressible, unmagnetized flows.