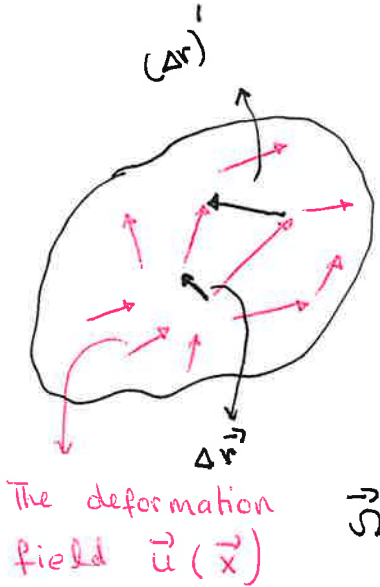


What we know:

a very small portion

- The most general deformation of a soft body can be decomposed into

- A translation
- A rotation
- A compression/extension in 3 orthogonal directions



$$(\Delta \vec{r})' = (\Delta \vec{r}) + \vec{u}(\vec{x}) + \frac{1}{2} \vec{\Omega} \times (\Delta \vec{r}) + S_{\alpha\beta} (\Delta r)_{\beta}$$

$$\vec{\Omega} = \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ u_x & u_y & u_z \end{pmatrix}$$

$$S_{\alpha\beta} = \frac{1}{2} \left(\partial_{\alpha} u_{\beta} + \partial_{\beta} u_{\alpha} \right) \quad \text{symmetric}$$

The strain tensor/matrix

$$\text{or } \Omega_{\alpha} = \epsilon_{\alpha\beta\gamma} \partial_{\beta} u_{\gamma}$$

Anti-symmetric

Quotient Rule :

- Strain is a rank two tensor
- $\vec{\Omega}$ is a rank one tensor (a vector)

If $A_{\alpha} = T_{\alpha\beta} B_{\beta}$ where $\vec{A}$ and $\vec{B}$ are vectors, then $T_{\alpha\beta}$ is a second rank tensor

- Forces : Gravity a "volume force".

Forces in soft solids are short range.

Crucial assumption there are no

"bare" ~~electrostatic~~ electrostatic forces.

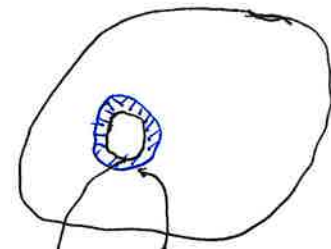
There are, of course, screened electrostatic forces.

As forces act through surface it is meaningful to define force - per - unit - area

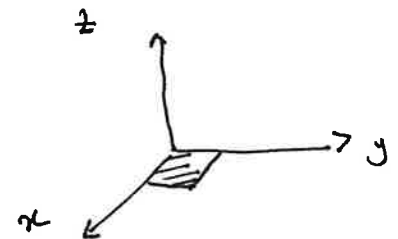
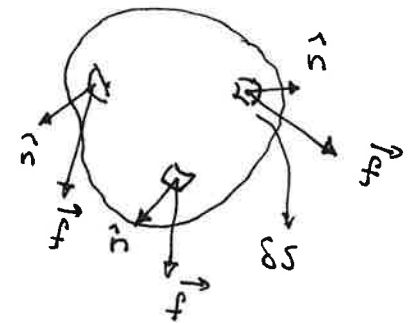
$$\left(\begin{array}{c} \text{something like} \\ \text{pressure} \end{array} \right) = \lim_{\delta S \rightarrow 0} \frac{\text{force over } \delta S}{(\delta S)}$$

Area is typically a vector.

Infinitesimal area : $\hat{n} \delta S$



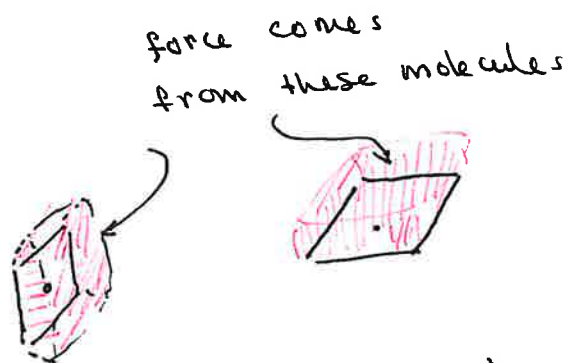
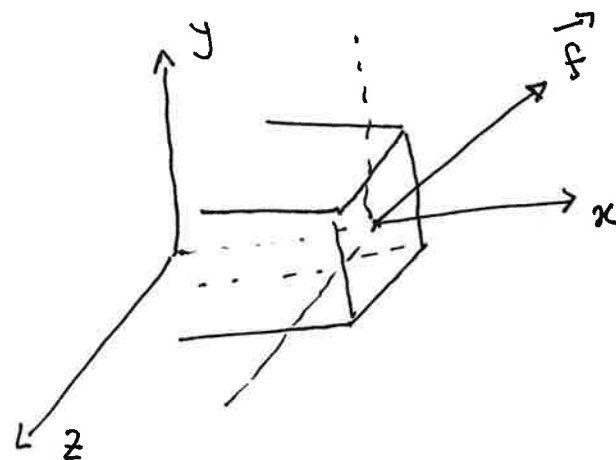
force outside this shell is zero.



$$\begin{aligned} \vec{\delta S} &= \hat{i} \delta x \times \hat{j} \delta y \\ &= \hat{z} (\delta x \delta y) \\ &= \hat{n} \delta S \end{aligned}$$

3

At the same point $\vec{x}$ if I consider surfaces with different orientation different forces act on them.



- So to specify the forces in a solid body I must specify
1. where ($\vec{x}$)
 2. which area ($\vec{n}$ & δS)
 3. what is the force ($\vec{f}$)

$$\begin{aligned}
 (\Delta y \Delta z) \sigma_{xx} \left(x + \frac{\Delta x}{2}, y, z \right) &= f_x \left(x + \frac{\Delta x}{2}, y, z \right) \\
 (\Delta x \Delta z) \sigma_{yx} \left(x, y + \frac{\Delta y}{2}, z \right) &= f_y \left(x, y + \frac{\Delta y}{2}, z \right) \\
 (\Delta x \Delta y) \sigma_{zx} \left(x, y, z + \frac{\Delta z}{2} \right) &= f_z \left(x, y, z + \frac{\Delta z}{2} \right)
 \end{aligned}$$

$\sigma_{\alpha\beta}(\vec{x})$
 component of force
 component of area

What is the net force on a small volume element?

$$\Delta V = \Delta x \Delta y \Delta z$$

$$f_x = \sigma_{xp} n_p (\Delta S)$$

$$f_y(x + \frac{\Delta x}{2}, y, z) + f_y(x - \frac{\Delta x}{2}, y, z)$$

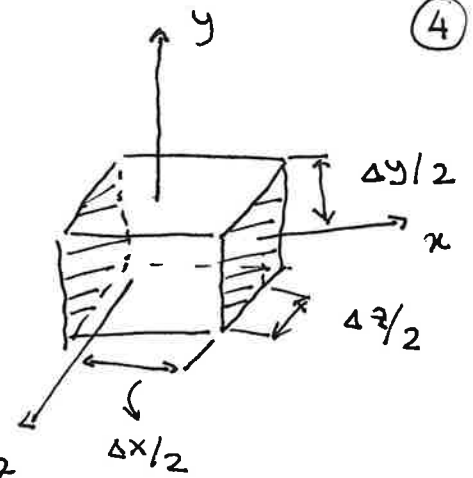
$$= \left[\sigma_{yx}(x + \frac{\Delta x}{2}, y, z) - \sigma_{yx}(x - \frac{\Delta x}{2}, y, z) \right] \Delta y \Delta z$$

y component
on the faces
⊥ to the x axis

because the outward normal $\hat{n}$ has opposite sign

$$= \left[\sigma_{yx}(x, y, z) + \frac{\partial \sigma_{yx}}{\partial x} \Delta x + \text{h.o.t} - \left(\sigma_{yx}(x, y, z) + \frac{\partial \sigma_{yx}}{\partial x} \left(-\frac{\Delta x}{2} \right) + \text{h.o.t} \right) \right] \Delta y \Delta z$$

$$= \frac{\partial \sigma_{yx}}{\partial x} \Delta x \Delta y \Delta z = \frac{\partial \sigma_{yx}}{\partial x} \Delta V$$



The net force along y direction summed over all the faces

$$F_y = \left(\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} \right) \Delta x \Delta y \Delta z$$

$$F_2 = \sum_{\beta=1}^3 \frac{\partial \sigma_{2\beta}}{\partial x_\beta} (\Delta V) = \frac{\partial \sigma_{2\beta}}{\partial x_\beta} (\Delta V)$$

$$\Rightarrow \quad \frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} \quad \Rightarrow \quad \boxed{F_\alpha = \frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} \Delta V}$$

$$= \partial_\beta \sigma_{\alpha\beta} \Delta V$$

Notes

(a) stress has the same dimension a pressure. In fact usual pressure is just the diagonal components of stress.

(b) stress is a 2nd rank tensor.

⑥

③ Newton's law for a small volume element in a soft body

$$\rho \frac{d\mathbf{u}}{dt} = \partial_j \sigma_{ij} + \text{volume forces } (\rho \mathbf{g})$$

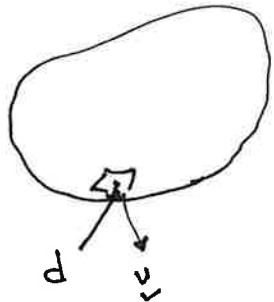
velocity → density →

For static deformation

$$\mathbf{v} = 0$$

$$\partial_j \sigma_{ij} = 0$$

Egn. of mechanical equilibrium



④

At the boundary of a soft body let there be an external force $\vec{P}$. Then at the surface $\vec{n} \cdot \vec{S} = 0$

internal forces

$$\Rightarrow \boxed{\vec{P} + \sigma_{ij} \vec{n}_j = 0}$$

⑦

Equilibrium means not only force - balance but also

⑧

torque balance.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau_x = \epsilon_{x\beta\gamma} x_\beta F_\gamma = \epsilon_{x\beta\gamma} x_\beta \partial_\mu \gamma_\mu \Delta V$$

$$\int \epsilon_{x\beta\gamma} x_\beta \partial_\mu \gamma_\mu \Delta V = \text{Not torque}$$

$$= \int_V \left[\partial_\mu (x_\beta \gamma_\mu) - \gamma_\mu \partial_\mu x_\beta \right] \Delta V$$

$$= \epsilon_{x\beta\gamma} \left[\oint_S x_\beta \gamma_\mu n_\mu ds - \int \gamma_\mu \partial_\mu x_\beta \Delta V \right]$$

$$= - \int \epsilon_{x\beta\gamma} \gamma_\beta \Delta V \rightarrow \text{must be zero.}$$

if no ext. forces



$$\epsilon_{\alpha\beta\gamma} \sigma_{\alpha\beta} = \sum_{\beta,\gamma=1}^3 \epsilon_{1\beta\gamma} \sigma_{1\beta\gamma} \quad \text{for } \alpha = 1$$

$$= \epsilon_{123} \sigma_{23} + \epsilon_{132} \sigma_{32}$$

$$= \sigma_{23} - \sigma_{32}$$

$$\boxed{\sigma_{\alpha\beta} = \sigma_{\beta\alpha}}$$

In general

Summary: Egn. of motion of a small volume element ΔV is

$$\rho \frac{d u_{\alpha}}{dt} = \partial_{\beta} \sigma_{\alpha\beta}$$

To think In elasticity we think about force through a surface as the physically important quantity.

$\sigma_{\alpha\beta}$ is a symmetric tensor

$$\vec{f}^{\alpha} = \sigma_{\alpha\beta} \vec{n}_{\beta} dS$$

where $\sigma_{\alpha\beta}$ is the stress tensor. defined as the force on an infinitesimal surface $\vec{n} dS$ is

stress must be a symmetric tensor.

all others are zero