

(1)

"Always step back before a long jump." - Unknown

What is a vector?

1. A stick with an arrowhead that obeys certain algebra.
2. A vector is a physical quantity that "behaves" like the position vector.

~~What do we~~

The duck theorem: If it looks like a duck, swims like a duck and quacks like a duck then it is probably a duck.

What do we precisely mean by "behaves"?

How it changes under coordinate transformation

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coordinates and their transformations:

Physical laws must be the same in all coordinate systems.

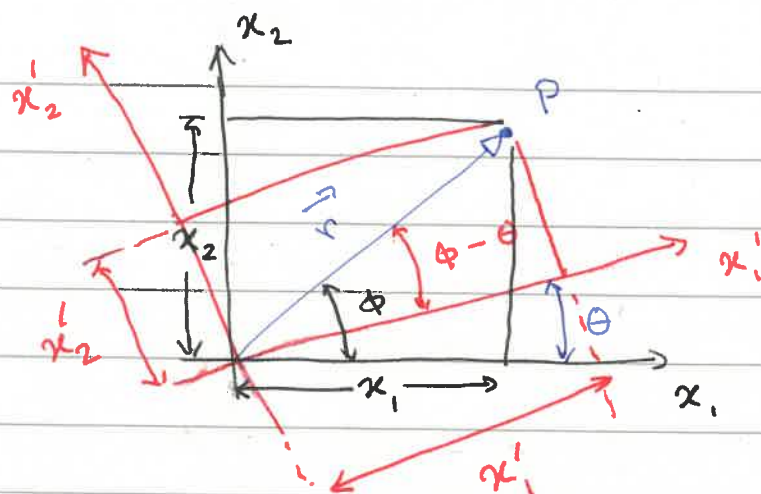
A physical law is a mathematical equation.

A mathematical equation must ~~transform~~ change under coordinate transformation in very specific ways.

A. Scalar : They do not change under coordinate transformation.

B. Vectors : ~~They~~ The position vector changes in a very specific way. Every vector must change in exactly the same way.

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$$\vec{OP} = \vec{r} = (x_1, x_2) = (x'_1, x'_2)$$

$$x_1 = r \cos \phi$$

$$x'_1 = r \cos (\phi - \theta)$$

$$= r (\cos \phi \cos \theta + \sin \phi \sin \theta)$$

~~$$= r \cos \phi \cos \theta + r \sin \phi \sin \theta$$~~

$$= r \cos \phi \cos \theta + r \sin \phi \sin \theta$$

$$\Rightarrow \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\boxed{x'_\alpha = a_{\alpha\beta} x_\beta}$$

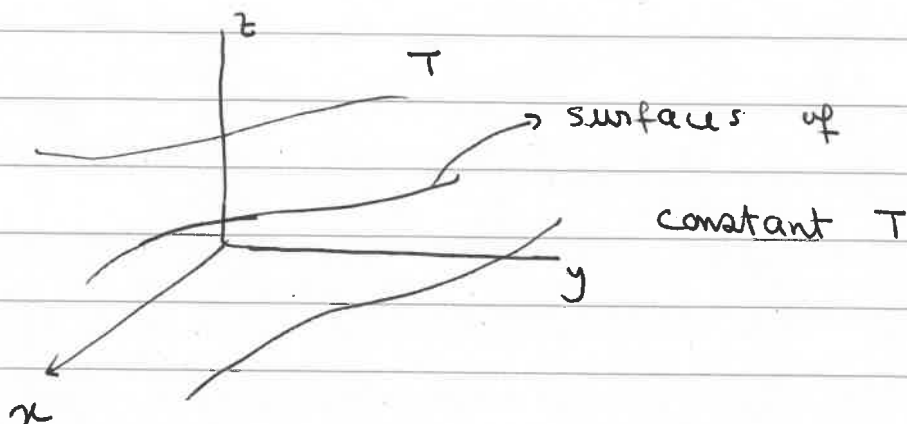
Any vector $\vec{A}$ with components $\{A_\alpha\}$ and $\{A'_\alpha\}$ in coordinates $\{x\}$ and $\{x'\}$ must satisfy

$$\boxed{A'_\alpha = a_{\alpha\beta} A_\beta}$$

This is the definition of a vector

(4.)

Scalar fields: example temperature, density

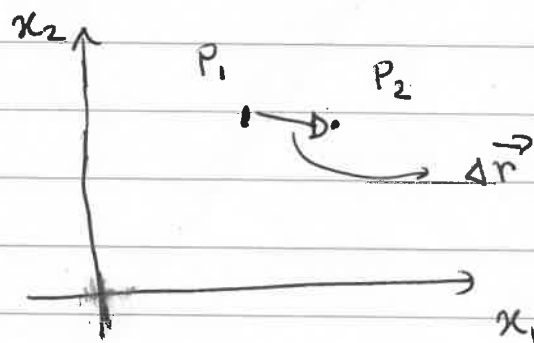


Derivative of scalar fields

$$\vec{\nabla} T \equiv \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right)$$

What kind of physical quantity is $\vec{\nabla} T$?

Consider the problem in 2-d



The difference in T between P_1 and P_2 is

$$\begin{aligned} \Delta T &= \frac{\partial T}{\partial x_1} \Delta x_1 + \frac{\partial T}{\partial x_2} \Delta x_2 + \sqrt{\frac{\partial T}{\partial x}} \\ &= \frac{\partial T}{\partial x'_1} \Delta x'_1 + \frac{\partial T}{\partial x'_2} \Delta x'_2 \end{aligned}$$

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$$\Delta T = \frac{\partial T}{\partial x_1} (\Delta x'_1 \cos \theta - \cancel{\Delta y'_1 \sin \theta}) \Delta x'_2 \sin \theta + \frac{\partial T}{\partial x_2} (\Delta x'_1 \sin \theta + \cancel{\Delta y'_1} \cancel{\Delta x'_2} \Delta x'_2 \cos \theta)$$

$$= \left(\frac{\partial T}{\partial x_1} \cos \theta + \frac{\partial T}{\partial x_2} \sin \theta \right) \Delta x'_1 + \left(-\frac{\partial T}{\partial x_1} \sin \theta + \frac{\partial T}{\partial x_2} \cos \theta \right) \Delta x'_2$$

$$\Rightarrow \begin{pmatrix} \frac{\partial T}{\partial x'_1} \\ \frac{\partial T}{\partial x'_2} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \partial T / \partial x_1 \\ \partial T / \partial x_2 \end{pmatrix}$$

$$\Rightarrow (\vec{\nabla} T)' = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \vec{\nabla} T \end{pmatrix}$$

Hence by the "deck theorem" $\vec{\nabla} T$ must be a vector.

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Now let us find an even simpler proof
(actually the same proof)

$$\Delta T = \frac{\partial T}{\partial x_1} \Delta x_1 + \frac{\partial T}{\partial x_2} \Delta x_2 + \frac{\partial T}{\partial x_3} \Delta x_3$$

$$= \frac{\partial T}{\partial x_\mu} \Delta x_\mu$$

$$(\Delta T) = (\vec{\nabla} T) \cdot (\Delta \vec{x})$$

$\uparrow$
 Temp. diff.
 must be a
 scalar

$\uparrow$
 scalar
 product

must be a vector too.

If ~~as~~ you can form a scalar product ~~then~~
~~you can figure out~~ with a known
 quantity then you can figure out the
 transformation property of the old quantity

Example

$$(\text{work done}) = \vec{F} \cdot \Delta \vec{x}$$

$\uparrow$
 scalar

$\uparrow$
 vector

must be a vector
too.

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The gradient:

$$\vec{\nabla} \equiv \begin{cases} \text{A differential operator} \\ \text{A vector} \end{cases}$$

$$\vec{\nabla} \cdot \vec{A} \leftarrow \text{vector field}$$

↑
vector differential operator

$$= \partial_x A_x + \partial_y A_y + \partial_z A_z$$

The divergence

$$= \partial_\mu A_\mu \quad \text{A scalar.}$$

$$\vec{\nabla} \times \vec{A} = \epsilon_{\beta\gamma} \partial_\beta A_\gamma$$

The curl.

A vector

Second operation of gradients:

~~$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A})$$~~
$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A})$$

~~Use the dual character of $\vec{\nabla}$~~ Use the dual character of $\vec{\nabla}$

Remember:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\left. \begin{array}{l} \vec{A} \rightarrow \vec{\nabla} \\ \vec{B} \rightarrow \vec{\nabla} \\ \vec{C} \rightarrow \vec{A} \end{array} \right\} \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - (\vec{\nabla} \cdot \vec{\nabla}) \vec{A}$$

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what is $\vec{\nabla} \cdot \vec{\nabla}$?

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The Laplacian

$$\equiv \nabla^2$$

$$\Rightarrow \vec{\nabla} \times \vec{\nabla} \times \vec{A} = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

Evaluate $\vec{\nabla} \times (\vec{A} \times \vec{B})$

use operator properly (product rule)

$$\vec{\nabla}_A \times (\vec{A} \times \vec{B}) \quad \leftarrow \quad \vec{\nabla}_B \times (\vec{A} \times \vec{B})$$

$$\begin{aligned} \vec{A} &\rightarrow \vec{\nabla}_A \\ \vec{B} &\rightarrow \vec{A} \\ \vec{C} &\rightarrow \vec{B} \end{aligned} \quad \text{for } \vec{A} \times (\vec{B} \times \vec{C})$$

similarly
complete this
half.

$$= \vec{A} (\vec{\nabla}_A \cdot \vec{B}) - \vec{B} (\vec{\nabla}_A \cdot \vec{A})$$

$$= (\vec{B} \cdot \vec{\nabla}) \vec{A} - (\vec{\nabla} \cdot \vec{A}) \vec{B}$$

Partial summary :

1. A vector is a quantity whose components transform like components of the position vector under rotation of coordinates

Under the coordinate transformation

$$x'_\alpha = a_{\alpha\beta} x_\beta$$

- The components of a vector transform as

$$A'_\alpha = a_{\alpha\beta} A_\beta$$

~~Defined~~

2. The operator

$$\begin{aligned}\vec{\nabla} &\equiv (\partial_x, \partial_y, \partial_z) \\ &\equiv (\partial_1, \partial_2, \partial_3)\end{aligned}$$

Is ~~are~~ a vector differential operator.

Tensors :

~~A tensor of rank k is~~

A tensor of rank k is a quantity

that transforms in the following manner

under coordinate transformation :

$$\underbrace{T'_{\alpha\beta\gamma\dots}}_k = \underbrace{a_{\alpha\mu} a_{\beta\nu} a_{\gamma\delta} \dots}_k \underbrace{T_{\mu\nu\delta\dots}}_k$$

Example : Strain

$$S_{\alpha\beta} = \frac{1}{2} \left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right)$$

From Helmholtz's theorem

$$\underbrace{(\Delta r)'_\alpha}_{\uparrow} = \dots + S_{\alpha\beta} (\Delta r)_\beta$$

(This prime has nothing
to do with coordinate
transformation but
actual physical deformation)

(2)

With a change of notation

$$d_\alpha = s_{\alpha\beta} r_\beta$$

Then $\vec{d}$ and $\vec{r}$ are vectors

So in a new coordinate system

$$d'_\alpha = a_{\alpha\beta} d_\beta$$

~~$$d'_\alpha = a_{\alpha\beta} d_\beta$$~~

$$= a_{\alpha\beta} s_{\beta\mu} r_\mu$$

$$d'_\alpha = s'_{\alpha\beta} r'_\beta$$

Then I need to invert

$$r'_\alpha = a_{\alpha\beta} r_\beta$$

How do I do that?

Notice that

$r_{\beta\beta}$ is a scalar

$r'_{\alpha\alpha}$ is a scalar

$$\Rightarrow r'_{\alpha\alpha} = a_{\alpha\beta} r_\beta a_{\alpha\mu} r_\mu$$

$$= a_{\alpha\beta} a_{\alpha\mu} r_{\beta\mu} = r_{\mu\mu}$$

$$\Rightarrow a_{\alpha\beta} a_{\alpha\mu} = \delta_{\beta\mu}$$

Then consider again

$$r'_\alpha = a_{\alpha\beta} r_\beta$$

$$r'_\alpha a_{\alpha\mu} = a_{\alpha\beta} a_{\alpha\mu} r_\beta$$

$$= \delta_{\beta\mu} r_\beta = r_\mu$$

$$\Rightarrow r_\mu = r'_\alpha a_{\alpha\mu}$$

The inverse of a is its transpose.

can substituting back

$$a'_\alpha = a_{\alpha\beta} s_{\beta\mu} r_\mu$$

$$= a_{\alpha\beta} s_{\beta\mu} a_{\delta\mu} r'_\delta = s'_{\alpha\delta} r'_\delta$$

$$\Rightarrow \boxed{s'_{\alpha\delta} = a_{\alpha\beta} a_{\delta\mu} s_{\beta\mu}}$$

$s_{\alpha\delta}$ is a second rank tensor.