

①

Helmholtz's theorem

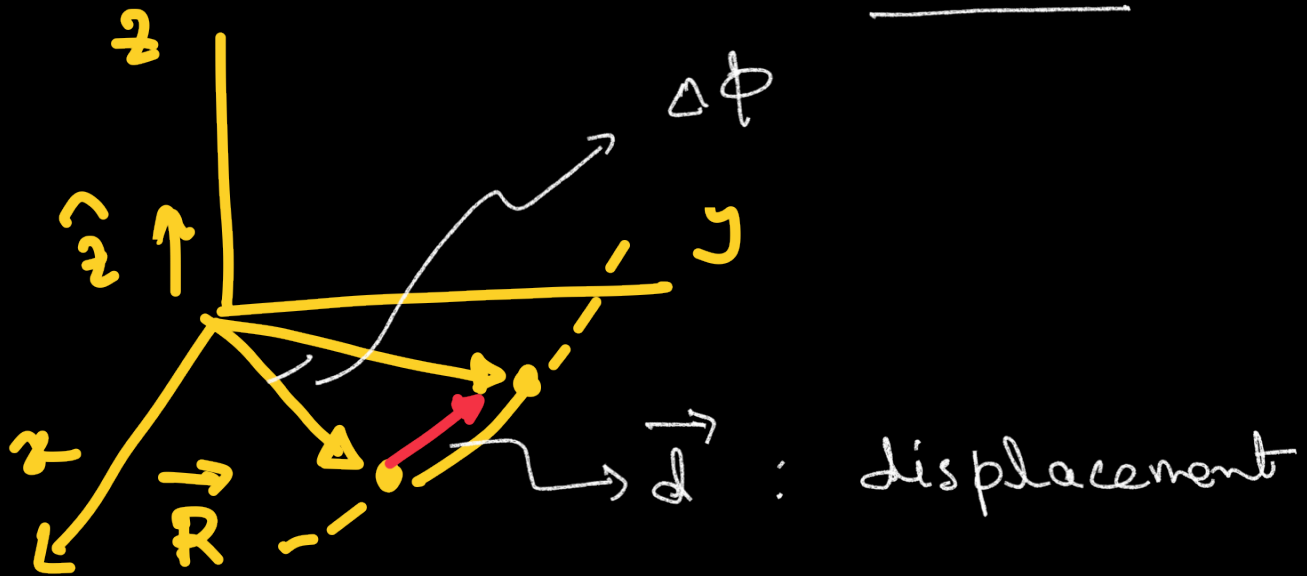
The most general displacement of a sufficiently small element of a deformable body is given by

- A translation
- A rotation
- An extension (contraction) in three mutually orthogonal directions.

2

Intutions :

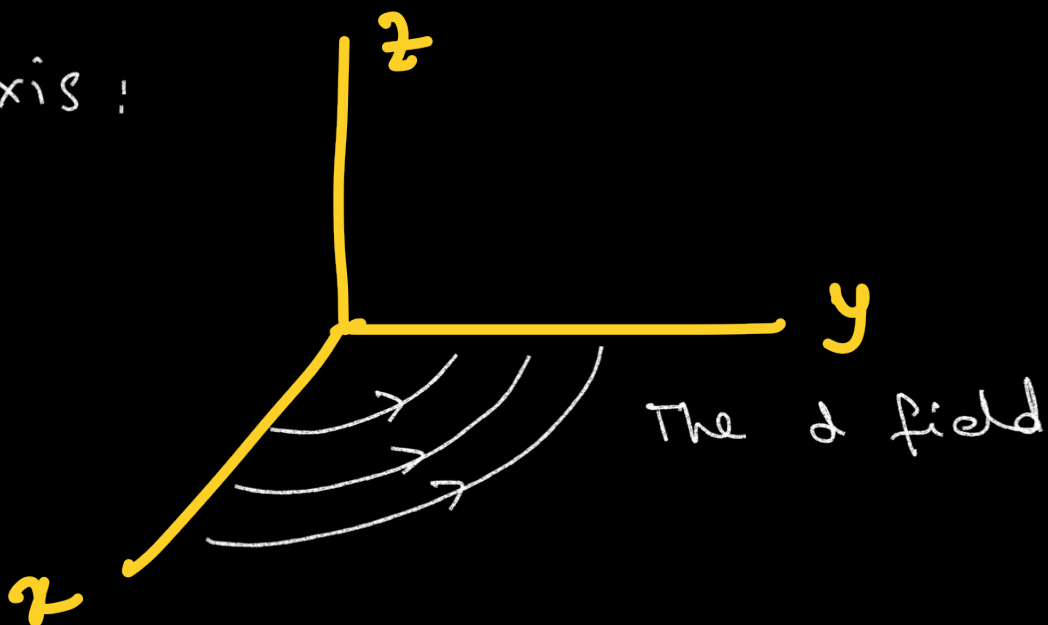
case 1



$$\vec{d} = R \Delta\phi (\hat{z} \times \hat{R})$$

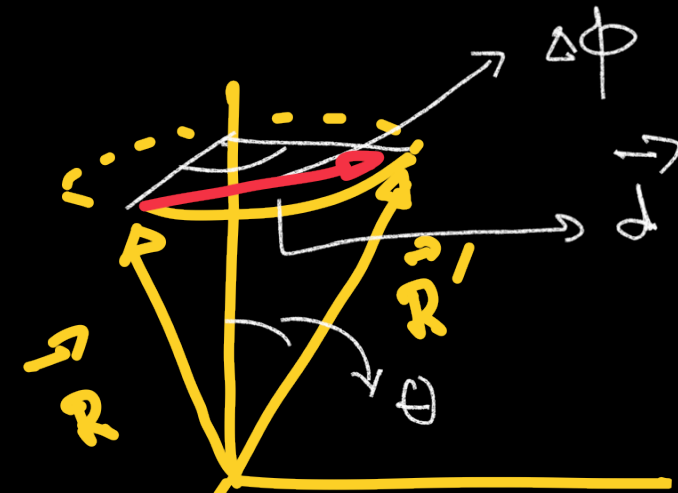
correct, when $\Delta\phi$ is small.

For pure rotation about z axis:



(3)

Generalise !

case 2

$$\vec{d} = R \sin \theta \Delta \phi \hat{z} \times \hat{R}$$

true when $\Delta \phi$ is small

$\Rightarrow$ For a small rotation about any axis

case 3

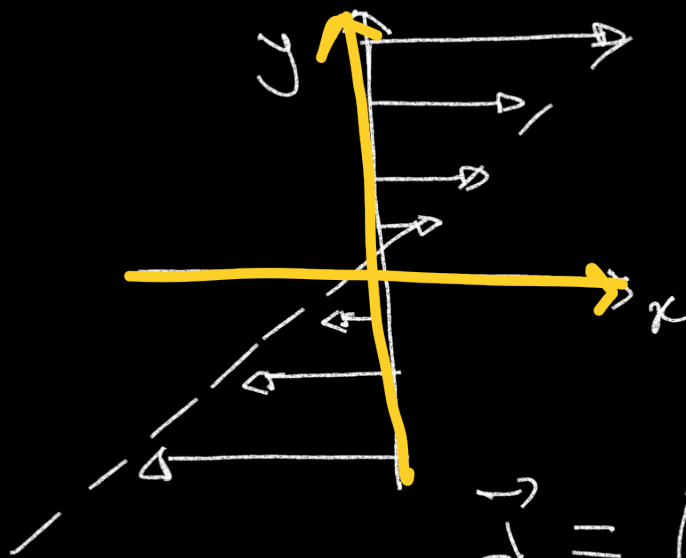
$$\vec{d} = (\hat{n} \times \vec{R}) \Delta \phi$$

unit vector
along axis of
rotation

$$\Rightarrow d_{\alpha} = \epsilon_{\alpha\beta\gamma} n_{\beta} R_{\gamma} \Delta \phi$$

Case 4

(4)

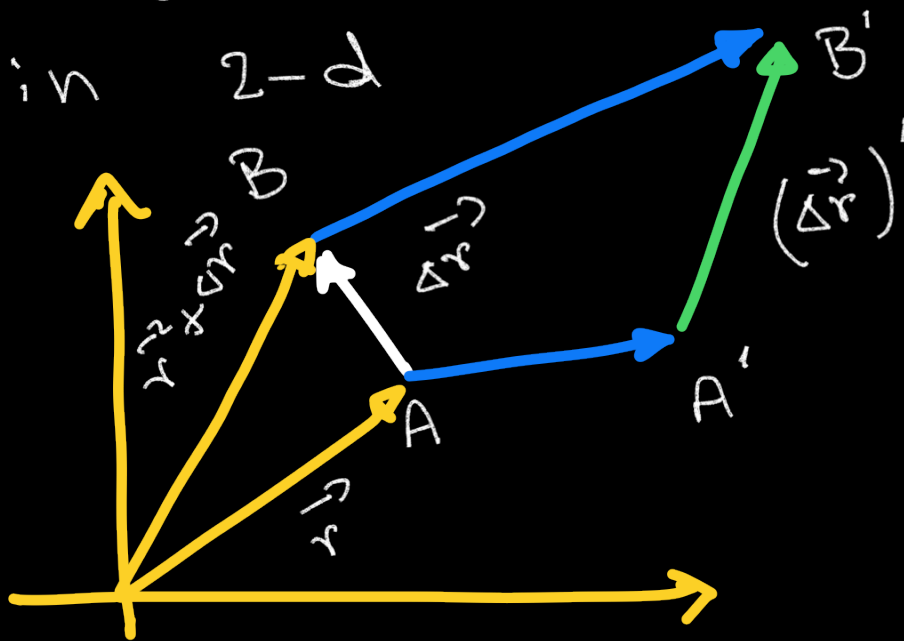


$$d_y = 0$$

$$d_x = sy$$

$$\vec{d} = (sy, 0, 0)$$

A general displacement
in 2-d



Before deformation: A: $\vec{r}$
B: $\vec{r} + \Delta \vec{r}$
 $\vec{AB}: \Delta \vec{r}$

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Deformation

$$\begin{array}{ccc}
 A & \rightarrow & A' \\
 B & \rightarrow & B' \\
 \overrightarrow{AB} & \longrightarrow & \overrightarrow{A'B'}
 \end{array}$$

The deformation is given by a vector field $\vec{u}$

$$\vec{u}(\vec{r}) = \overrightarrow{AA'}$$

$$\vec{u}(\vec{r} + \Delta\vec{r}) = \overrightarrow{BB'}$$

In general $\vec{u}(\vec{r}) = \vec{u}(x, y)$

$$\begin{aligned}
 (\Delta\vec{r}') &= \vec{r} + \Delta\vec{r} + \vec{u}(\vec{r} + \Delta\vec{r}) \\
 &\quad - [\vec{r} + \vec{u}(\vec{r})]
 \end{aligned}$$

$$\begin{aligned}
 &= \Delta\vec{r} + \vec{u}(\vec{r} + \Delta\vec{r}) \\
 &\quad - \vec{u}(\vec{r})
 \end{aligned}$$

$$\vec{u}(\vec{r} + \Delta\vec{r})$$

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$$= \begin{pmatrix} u_x(x+\Delta x, y+\Delta y) \\ u_y(x+\Delta x, y+\Delta y) \end{pmatrix}$$

$$u_x(x+\Delta x, y+\Delta y) = u_x(x, y) + \frac{\partial u_x}{\partial x} \Delta x$$

$$+ \frac{\partial u_x}{\partial y} \Delta y + \text{h.o.t.}$$

$$u_y(x+\Delta x, y+\Delta y) = u_y(x, y) + \frac{\partial u_y}{\partial x} \Delta x$$

$$+ \frac{\partial u_y}{\partial y} \Delta y + \text{h.o.t.}$$

$$\begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

⑦

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} & \frac{a_{12} + a_{21}}{2} \\ \frac{a_{12} + a_{21}}{2} & a_{22} \end{pmatrix} + \begin{pmatrix} 0 & \frac{a_{12} - a_{21}}{2} \\ \frac{a_{21} - a_{12}}{2} & 0 \end{pmatrix}$$

symmetric

anti-sym

Anti-symmetric part ⑧

$$\begin{aligned}
 \vec{d} &= \Delta\phi \left(\hat{z} \times \vec{r} \right) \\
 &= \Delta\phi \begin{pmatrix} \hat{x} y - \hat{y} x \\ 0 \\ x y - y x \end{pmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & 1 \\ x & y & 0 \end{vmatrix} \\
 &= \begin{pmatrix} 0 & \Delta\phi \\ -\Delta\phi & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
 \end{aligned}$$

Rotation $\rightarrow$ multiplication
(infinitesimal) by an anti-symmetric matrix

$$\Delta\phi = \frac{a_{12} - a_{21}}{2}$$

$$= \frac{1}{2} \left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right)$$

$$= \frac{1}{2} \Omega_z ; \quad \vec{\Omega} = \vec{\nabla} \times \vec{u}$$

$$\begin{aligned}
 (\Delta \vec{r})' &= \Delta \vec{r} + \vec{u}(\vec{r} + \Delta \vec{r}) - \vec{u}(\vec{r}) \quad (9) \\
 &= \Delta \vec{r} + \cancel{\vec{u}(\vec{r})} + \begin{pmatrix} \end{pmatrix} \begin{pmatrix} \end{pmatrix} + \begin{pmatrix} \end{pmatrix} \begin{pmatrix} \end{pmatrix} - \cancel{\vec{u}(\vec{r})}
 \end{aligned}$$

$\uparrow$
 (Δr)

$$= \Delta \vec{r} + \begin{pmatrix} \text{sym} \\ \text{mat.} \end{pmatrix} \begin{pmatrix} \end{pmatrix} + \begin{pmatrix} \text{anti-sym} \\ \text{mat} \end{pmatrix} \begin{pmatrix} \end{pmatrix}$$

$\uparrow$
 Δr

$$= \Delta \vec{r} + \begin{pmatrix} \text{sym} \\ \text{mat} \end{pmatrix} \begin{pmatrix} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & \Delta\phi \\ -\Delta\phi & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}}$$

This is a rotation
 by $\Delta\phi$ about z
 axis

(10)

sym matrix

$$\begin{pmatrix} a_{11} & \frac{a_{12} + a_{21}}{2} \\ \frac{a_{12} + a_{21}}{2} & a_{22} \end{pmatrix}$$

Any symmetric matrix can be diagonalized. This is a real

symmetric matrix $\Rightarrow$ Hermitian

$\Rightarrow$ Orthogonal eigenvectors

Real eigenvalues (λ_1, λ_2)

If $\lambda_1 > 0$ extension
 < 0 compression

Putting everything together

⑪

$$\begin{pmatrix} u_x(\vec{r} + \Delta\vec{r}) \\ u_y(\vec{r} + \Delta\vec{r}) \end{pmatrix} =$$

$$\begin{pmatrix} u_x(\vec{r}) \\ u_y(\vec{r}) \end{pmatrix} + \begin{pmatrix} 0 & \Delta\phi \\ -\Delta\phi & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

Translation

Rotation

$$+ \begin{pmatrix} \text{sym} \\ \text{matrix} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

compression / extension
along two orthogonal
directions

Generalize

(12)

$$\begin{aligned} u_{\alpha}(\vec{r} + \Delta\vec{r}) &= u_{\alpha}(\vec{r}) + \frac{1}{2} \vec{\Omega} \times \Delta\vec{r} \\ &\quad + \sum_{\alpha\beta} (\Delta r)_{\beta} \\ &= u_{\alpha}(\vec{r}) + \frac{1}{2} \epsilon_{\alpha\beta\gamma} \Omega_{\beta} (\Delta r)_{\gamma} \\ &\quad + \sum_{\alpha\beta} (\Delta r)_{\beta} \end{aligned}$$

$\vec{\Omega}$ and $\mathbb{S}$ are anti-symmetric and symmetric part of the gradient matrix $\frac{\partial u_{\alpha}}{\partial x_{\beta}}$

$$\left. \begin{aligned} \Omega_x &= \partial_y u_x - \partial_x u_y \\ \Omega_y &= \partial_z u_x - \partial_x u_z \\ \Omega_z &= \partial_x u_y - \partial_y u_x \end{aligned} \right| \vec{\Omega} = \vec{\nabla} \times \vec{u}$$

$$S_{\alpha\beta} = \left(\frac{\partial u_{\alpha}}{\partial x_{\beta}} + \frac{\partial u_{\beta}}{\partial x_{\alpha}} \right)$$

Strain matrix

Has two indices : Strain Tensor

Summary :

Helmholtz theorem.

The most general displacement of a deformable body can be decomposed into

an infinitesimal element

• translation

• rotation

• extension/contraction along 3 mutually $\perp^r$ directions.

strain matrix/tensor.

$$\frac{1}{2} \vec{\nabla} \times \vec{u}$$