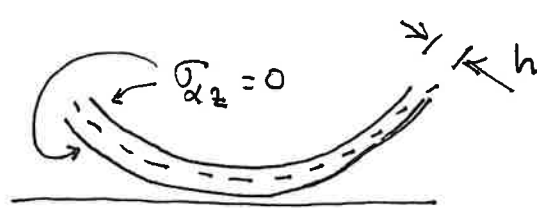


Plates

only bending



$$\vec{u} = (0, 0, w)$$

Nov 22

①

Method 1

- Integrate 3D free energy over the thickness of the plate to obtain a 2D free energy as a function of w alone

Key $\sigma_{xz} = 0$ everywhere

$$\mathcal{F}_{2D}^{\text{bend}} = \underbrace{\frac{Yh^3}{24(1-\sigma^2)}}_{B} \int (\partial_x^2 w + \partial_y^2 w)^2 dx dy$$

B : bending modulus

Method 2

- Landau like argument. $\mathcal{F}_{2D}^{\text{bend}}$ can depend only on $\kappa_{ij} = \frac{\partial^2 w}{\partial x_i \partial x_j}$

$$\text{Tr}(\kappa_{ij}) = \partial_x^2 w + \partial_y^2 w$$

It has two invariants:

The Gaussian curvature $K \leftrightarrow \text{Det}(\kappa_{ij})$

$$\mathcal{F}_{2D}^{\text{bend}} = B \int (\partial_x^2 w + \partial_y^2 w)^2 dx dy$$

$\int K ds =$ constant that depends only on topology of surface

only stretching: $\vec{u} = (u_1, u_2, 0)$

$$\mathcal{F}_{2D}^{\text{stretch}} = \frac{Y_{2D}}{2(1+\sigma_{2D})} \int \left[s_{ij}^2 + \frac{\sigma_{2D}}{1-2\sigma_{2D}} s_{kk}^2 \right] dx dy$$

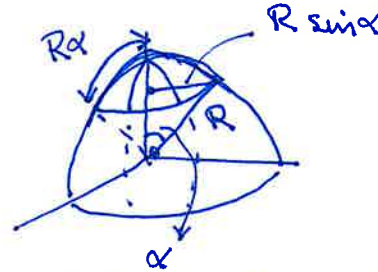
$$\mathcal{F}_{2D} = \mathcal{F}_{2D}^{\text{bend}} + \mathcal{F}_{2D}^{\text{stretch}}$$

$$s_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i)$$

$$Y_{2D} = \frac{Yh}{1-\sigma^2}, \quad \sigma_{2D} = \frac{\sigma}{1-\sigma}$$

Can we bend without stretch?

2



If it is bent without stretching then all lengths should map.

$$2\pi r = 2\pi R \sin \alpha$$

circumference

$$r = R \alpha$$

$$\Rightarrow \sin \alpha = \alpha \quad \text{true only when } \alpha \text{ is small.}$$

We can use a plane and bend it on the surface of a sphere without stretching only over a very small part of sphere. Not in general.

~~Bend~~ Bending must be typically accompanied by stretching.

$\mathcal{F}_{2D}^{\text{stretch}}$ should have a contribution from w . Of course, only from derivative of w , not w itself, $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$.

Only possible way

$$s_{ij} = \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j}$$

(3)

$$U_{\text{Stretch}}^{2D} = \frac{Y_{2D}}{2(1+\sigma_{2D})} \int \left[S_{ij}^2 + \frac{\sigma_{2D}}{1-2\sigma_{2D}} S_{kk}^2 \right] dx dy$$

$$S_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \underbrace{\frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j}}_{\text{biquadratic contribution to energy!}}$$

 $i, j = 1, 2$

Another way to see the same thing:

Gives a biquadratic contribution to energy!

$$(\Delta r)'_{\alpha} = (\Delta r)_{\alpha} + S_{\alpha\beta} (\Delta r)_{\beta} + \left(\text{other terms that don't matter} \right)$$

$$\Rightarrow (\Delta r)'_{\alpha} (\Delta r)'_{\alpha} = (\Delta r)_{\alpha} (\Delta r)_{\alpha} + 2 S_{\alpha\beta} (\Delta r)_{\beta} (\Delta r)_{\alpha}$$

$$+ S_{\alpha\beta} S_{\alpha\mu} (\Delta r)_{\beta} (\Delta r)_{\mu}$$

we ignored this before assuming strains are small.

$$S_{\alpha\beta} S_{\alpha\mu} = \underbrace{\frac{\partial u_1}{\partial x} \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \frac{\partial u_2}{\partial y}}_{\text{small strain, can be ignored.}} + \underbrace{\frac{\partial w}{\partial x} \frac{\partial w}{\partial y}}_{\text{Not necessarily small}}$$

$$U_{2D} = B \int \left(\frac{1}{R_1} + \frac{1}{R_2} \right)^2 dx_1 dx_2$$

Even for large bending

$$\Rightarrow \left| \begin{aligned} U_{2D} &= B \int \left(\frac{1}{R_1} + \frac{1}{R_2} \right)^2 dx_1 dx_2 \\ S_{ij} &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} + \frac{Y_{2D}}{2(1+\sigma_{2D})} \int \left(S_{ij}^2 + \frac{\sigma_{2D}}{1-2\sigma_{2D}} S_{kk}^2 \right) dx_1 dx_2 \end{aligned} \right.$$

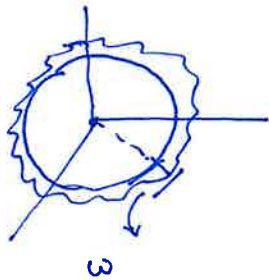
Important lesson:

- We still have the linear constitutive relation
stress $\propto$ strain
- But the problem is non-linear because energy is higher than quadratic in strain.
- The non-linearity is not coming from material property, it is coming because we have a low dimensional object embedded in higher dimension. It is of purely geometric origin.

From plate to shell

(5)

There is a significant change now. If ~~the~~ every point on the shell are sifted by a fixed amount - it is no longer translation, in the spherical gauge.



$$\text{strain} \sim \frac{w}{R}$$

$$s_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i) + \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} + \underbrace{\delta_{ij} \frac{w}{R}}_{\text{only for a shell}}$$

$$J_{2D}^{\text{shell}} = B \int \left[(\partial_x^2 + \partial_y^2) w \right]^2 dx dy + \frac{\chi_{2D}}{2(1 + \sigma_{2D})} \int \left(s_{ij}^2 + \frac{\sigma_{2D}}{1 - 2\sigma_{2D}} s_{kk}^2 \right) dx dy$$

$$\left(\text{terms} \propto w^2 \right) + \left(\propto u^2 \right) + \left(\sim w^4 \right) + \left(\sim w^3 \right) + \left(\text{product of } u \text{ and } (w^2, w) \right)$$

What do we do with all these information?

- what is the pressure at which a shell buckle?

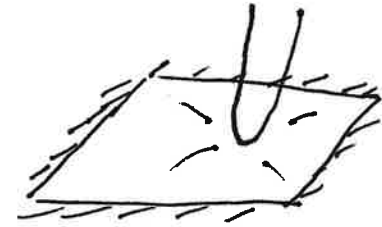
Ans. Not easy. often requires detail numerical calculation.

- Most engineering books deal with forces & moments not energy.

In principle, you can always solve these problems by looking for a minimum energy state. In practice, do what is best.

More problems in plates & shells.

1. I have a membrane fixed on a frame.



I poke it. what is the force-distance relationship?

2. What is the fluctuation spectrum of such a membrane?

3. What is the force-distance relationship of a shell?

