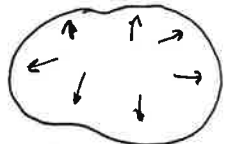


What we know:



$\vec{u}(\vec{r})$

deformation field

$$(\Delta \vec{r})' = (\Delta \vec{r}) + \vec{u}(\vec{r}) + \vec{\Omega} \times (\Delta \vec{r}) + S_{\alpha\beta} (\Delta r)_{\beta}$$

$\vec{\Omega} \times (\Delta \vec{r})$ ← anti symmetric
 $S_{\alpha\beta}$ ← symmetric

strain

Stress: force per unit area.

Hook's law

$$\sigma_{\alpha\beta} = E_{\alpha\beta\mu\nu} s_{\mu\nu}$$

material property

Inverse

$$= K s_{\alpha\beta} s_{\mu\mu} + 2\mu \left(s_{\alpha\beta} - \frac{1}{3} s_{\alpha\beta} s_{\mu\mu} \right)$$

$$= \frac{\gamma}{1+\sigma} \left(s_{\alpha\beta} + \frac{\sigma}{1-2\sigma} s_{\alpha\beta} s_{\mu\mu} \right)$$

$$\gamma s_{\alpha\beta} = (1+\sigma) \sigma_{\alpha\beta} - \sigma s_{\alpha\beta} \sigma_{\mu\mu}$$

Boundary condition:

$$(\sigma_{\alpha\beta}^o - \sigma_{\alpha\beta}^i) n_{\beta} = -P_{\alpha}$$

ext. force per unit area

$$\sigma_{\alpha\beta}^i n_{\beta} = 0$$

$$\text{for } \sigma_{\alpha\beta}^o = 0$$

$$\text{and } \vec{p} = 0$$

$$f_{\alpha}^{\text{ext}} + \partial_{\beta} \sigma_{\alpha\beta} = 0$$

Eqn. of static Equilibrium of a soft body

for homogeneous isotropic material

The equation of Elasticity

$$\frac{1}{1-2\sigma} \vec{\nabla} (\text{div } \vec{u}) + \nabla^2 \vec{u} = - \frac{2(1+\sigma)}{\gamma} \vec{f}$$

Navier - Cauchy Eqn.

Linear, 2nd order in space.

Free Energy Density

$$F = \frac{\gamma}{2(1+\sigma)} \left(s_{\alpha\beta}^2 + \frac{\sigma}{1-2\sigma} s_{\mu\mu}^2 \right)$$

Plates

To do:

While down "effective" eqn. of elasticity for a thin, effectively two dimensional, body.

Case 1

The body is bent but NOT stretched.

In general: $\vec{u} = (u_1, u_2, w)$

u_1, u_2 are constant.

$u_1^{(0)} = 0, u_2^{(0)} = 0$

On the neutral plane:

$w(x, y)$

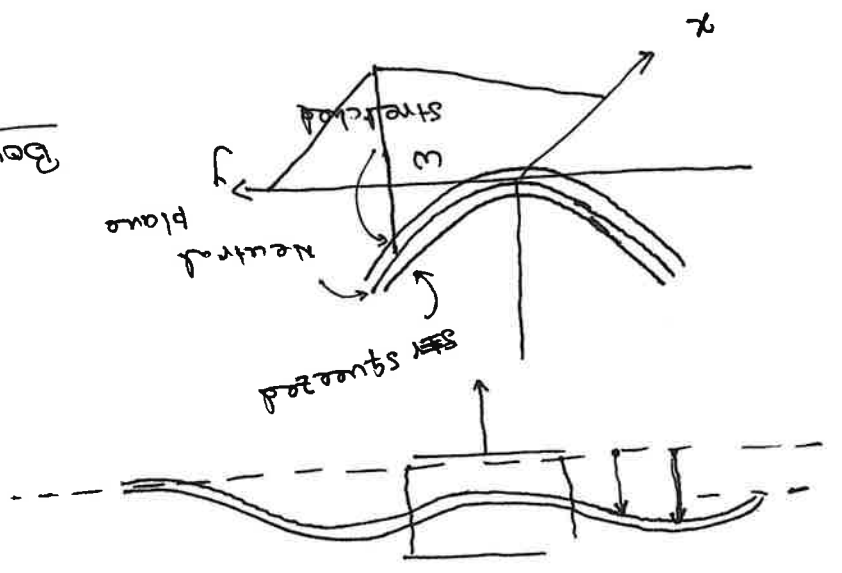
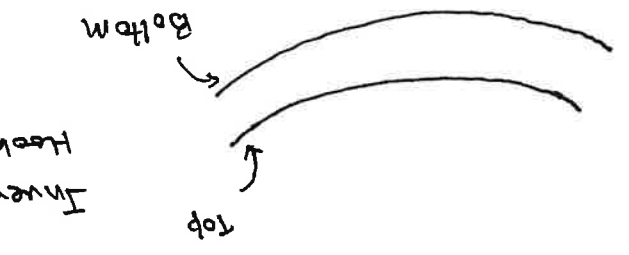
$a = x$	y	z
$\sigma_{xz} = 0$	$\sigma_{yz} = 0$	$\sigma_{zz} = 0$

at top { also zero
at bottom { inside

$\gamma_{xz} = (1 + \sigma) \sigma_{xz} = 0$

Inverse Hook's law

$\Rightarrow \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 0$



case 1

(3)

$$\Rightarrow u_x = -z \frac{\partial \omega}{\partial x}, \quad u_y = -z \frac{\partial \omega}{\partial y}$$

$$s_{xx} = \frac{\partial u_x}{\partial x} = -z \frac{\partial^2 \omega}{\partial x^2}$$

$$s_{xy} = -z \frac{\partial^2 \omega}{\partial x \partial y}$$

$$s_{yy} = -z \frac{\partial^2 \omega}{\partial y^2}$$

$$s_{yx} = s_{xy}$$

$$\left. \begin{array}{l} \nabla_{||}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{array} \right\}$$

Also

$$\text{Hook's law: } 0 = \sigma_{zz} = \frac{Y}{1+\sigma} \left(s_{zz} + \frac{\sigma}{1-2\sigma} s_{\mu\mu} \right)$$

$$\Rightarrow s_{zz} = -\frac{\sigma}{1-2\sigma} (s_{xx} + s_{yy} + s_{zz})$$

$$\Rightarrow s_{zz} = -\frac{\sigma}{1-\sigma} (s_{xx} + s_{yy}) = \frac{\sigma z}{1-\sigma} \nabla_{||}^2 \omega$$

$$F = \frac{Y}{2(1-\sigma)} \left(s_{\alpha\beta}^2 + \frac{\sigma}{1-2\sigma} s_{\mu\mu}^2 \right)$$

$$U_{3D} = \int dx dy dz F$$

$$= \frac{Y}{2(1-\sigma)} \int_{-h/2}^{+h/2} z^2 dz \int dx dy \left(\nabla_{||}^2 \omega \right)^2 + 2(1-\sigma) \left\{ \left(\frac{\partial^2 \omega}{\partial x \partial y} \right)^2 - \frac{\partial^2 \omega}{\partial x^2} \frac{\partial^2 \omega}{\partial y^2} \right\} \Bigg]$$

 U_{plate}

To obtain the equivalent of Navier-Cauchy Eqn. $\frac{\delta \mathcal{F}_{\text{plate}}}{\delta w} = 0$

(4)

$$\Rightarrow \boxed{\nabla_{||}^4 w = 0}$$

AMAZING

The other term conspire to drop off !!

$$\mathcal{A} = \int f(h, h', h'') dx$$

$$\frac{\delta \mathcal{A}}{\delta h} = \frac{\partial f}{\partial h} - \frac{d}{dx} \left(\frac{\partial f}{\partial h'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial h''} \right)$$

Landau Free energy of bent plate,

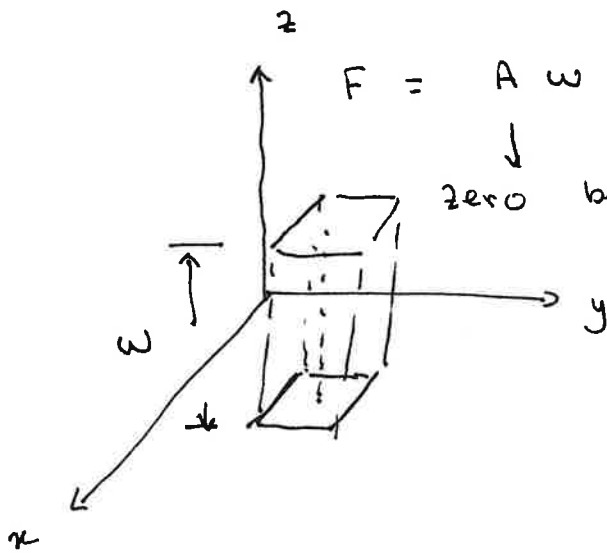
$$F = A w + B_k \frac{\partial w}{\partial x_k} + C_{ij} \frac{\partial w}{\partial x_i \partial x_j} + \text{h.o.t}$$

zero because translation

This is rotation

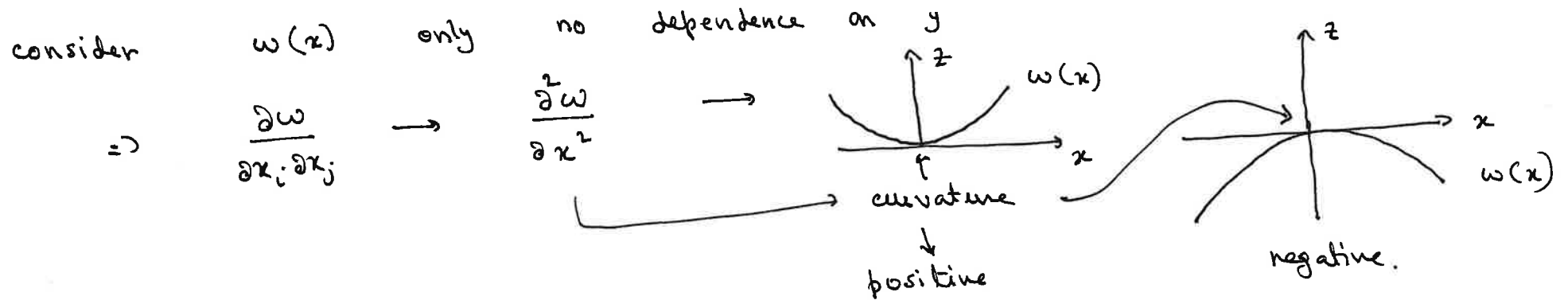
$$\frac{\partial w}{\partial x} = \frac{\partial u_z}{\partial x}, \quad \frac{\partial u_z}{\partial z} = 0 \quad \text{because we considered no stretching}$$

$\Rightarrow$ contributes only to anti-symmetric term. $\Rightarrow$ rotation.



$$\Rightarrow F = C_{ij} \frac{\partial^2 \omega}{\partial x_i \partial x_j}$$

cannot be correct.



Hence F can become negative.

Impossible because :

(a)

no thermodynamic equilibrium

(b)

If there is no difference between top and bottom, then F should not depend on the sign of $\partial^2 \omega / \partial x^2$

$$\Rightarrow F = \left(\begin{array}{l} \text{some quadratic} \\ \text{function of } \partial^2 \omega / \partial x_i \partial x_j \end{array} \right)$$

$$\frac{\partial^2 \omega}{\partial x_i \partial x_j}$$

$$\begin{pmatrix} \frac{\partial^2 \omega}{\partial x_1^2} & \frac{\partial^2 \omega}{\partial x_1 \partial x_2} \\ \frac{\partial^2 \omega}{\partial x_1 \partial x_2} & \frac{\partial^2 \omega}{\partial x_2^2} \end{pmatrix}$$

$$\text{Trace} =$$

$$\frac{\partial^2 \omega}{\partial x_1^2} + \frac{\partial^2 \omega}{\partial x_2^2}$$

$$\Delta^2 \omega$$

$$= \frac{\partial^2 \omega}{\partial x_1^2} \frac{\partial^2 \omega}{\partial x_2^2} - \frac{\partial^2 \omega}{\partial x_1 \partial x_2} \frac{\partial^2 \omega}{\partial x_2 \partial x_1}$$

$$\text{Det} =$$

A matrix

has two invariants

$$\text{Def. } (Tr)^2, \text{ Det.}$$

Quadratic

$$F = C \left(Tr \frac{\partial^2 \omega}{\partial x_i \partial x_j} \right)^2 + 1$$

and

$$g_{\text{plak}} = \int dx dy F$$

The same as what we obtained before.

$$\text{Det} \left(\frac{\partial^2 \omega}{\partial x_i \partial x_j} \right)$$

⑦

Differential Geometry

R : radius of curvature

$$\kappa \equiv \text{curvature} = \frac{1}{R}$$

$\vec{T} \equiv$ unit tangent
 $\vec{r}(x(s), y(s))$

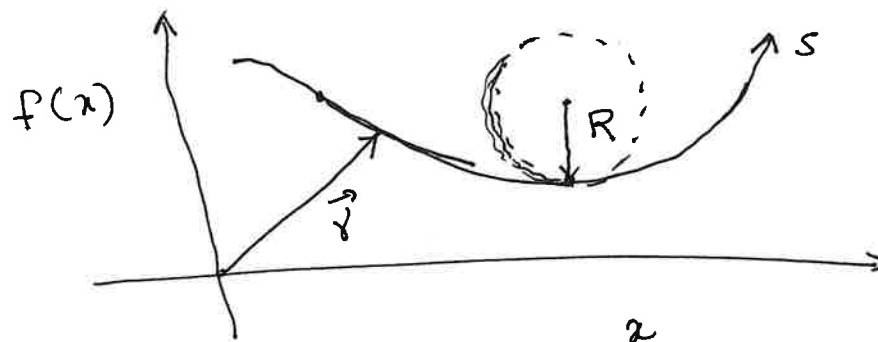
Parametric curve

$$\vec{T} = (x', y')$$

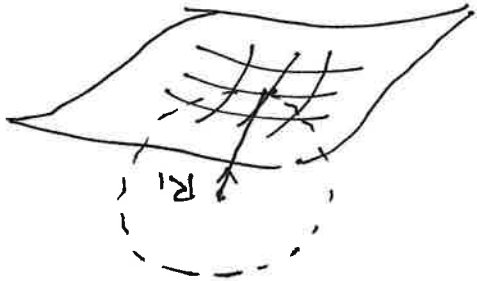
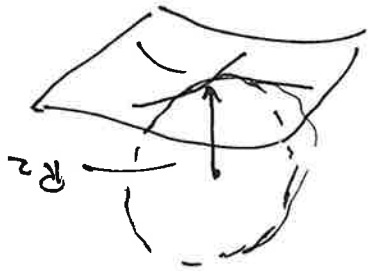
$$\kappa = \frac{d\vec{T}}{ds} = \frac{f''}{(1 + (f')^2)^{3/2}} \approx f'' \quad \text{if } f' \ll 1$$

Thus:

$$\text{curvature} \approx \frac{d^2 f}{dx^2}$$



8

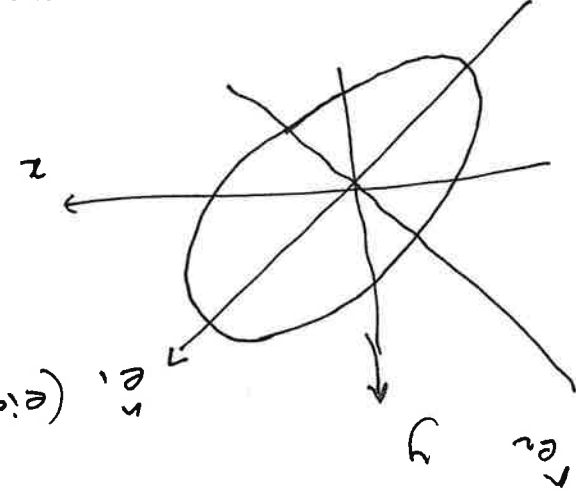


Diagonalize (c_1, c_2) eigenvalues

principal curvatures.

e_1, e_2 (eigenvectors)

Quadratic form of the curvature matrix.



Mean curvature

$$2H = (c_1 + c_2)$$

curvature

Gaussian

$$K = \det K_{ij} = c_1 c_2$$

$$U_{\text{plate}} = \int dx dy \left(\underbrace{4B H^2}_{\substack{\text{Bending} \\ \text{modulus}}} + \Lambda K \right)$$

Gauss-Bonnet theorem: For a surface the integral of the Gaussian curvature is a quantity that depends only on the topology of the surface.

$$\int dx dy K = \text{constant}$$

$$\Rightarrow U_{\text{plate}} = B \int dx dy (\nabla_{||}^2 \omega)^2 \quad \text{small bending}$$

$$= B \int dx dy H^2 \quad \text{Any bending}$$

Assumptions

1. small strain
2. ~~Bending~~ & we can bend without stretch.

$$B = \frac{\gamma h^3}{24(1-\sigma^2)}$$

$$\vec{u} = (u_1, u_2, 0)$$

(10)

only stretching

$$s_{xz} = \text{wave } \sigma \quad \sigma_{xz} = 0 = \sigma_{yz}; \quad s_{zz} = -\frac{\sigma}{1-2\sigma} (\sigma_{xx} + \sigma_{yy})$$

$$\Rightarrow s_{\mu\mu} = \frac{1-2\sigma}{1-\sigma} (\sigma_{xx} + \sigma_{yy}) \quad , \quad s_{xz} = s_{yz} = 0$$

$$U_{3D} = \int \left(\frac{1}{2} \lambda s_{\mu\mu}^2 + \mu s_{\beta\beta}^2 \right) dx dy dz$$

$$= \int \left[\frac{1}{2} \lambda \left(\frac{1-2\sigma}{1-\sigma} \right)^2 (\sigma_{xx} + \sigma_{yy})^2 + \mu s_{xy}^2 \right] dx dy$$

$$= \int \left[() (\sigma_{xx} + \sigma_{yy})^2 + () s_{xy}^2 \right] dx dy \int_{-h/2}^{+h/2} dz$$

$$U_{2D} = \int dx dy [\quad] h$$

$$= \frac{Y_{2D}}{2(1-\sigma_{2D})} \int \left(s_{ij}^2 + \frac{\sigma_{2D}}{1-2\sigma_{2D}} s_{kk}^2 \right) dx dy$$

$$Y_{2D} = \frac{Yh}{1-\sigma^2}$$

$$\sigma_{2D} = \frac{\sigma}{1-\sigma}$$

Summary

Assumption :

- Bending and stretching are considered independent
- Strain is considered small.

$$\vec{u} = (u_1(x, y), u_2(x, y), w(x, y))$$

Equation of 3D elasticity
can be used to find G.P.N.
of 2D elasticity.

Integrate the Free Energy over
the thickness of the plate

$$\mathcal{F}_{\text{plate}} = \mathcal{F}_{\text{bend}} + \mathcal{F}_{\text{stretch}}$$

$$\mathcal{F}_{\text{bend}} = B \int (\nabla_{11}^2 w)^2 dx dy$$

$$\mathcal{F}_{\text{stretch}} = \frac{Y_{2D}}{2(1-\sigma_{2D})} \int \left(S_{ij}^2 + \frac{\sigma_{2D}}{1-2\sigma_{2D}} S_{kk}^2 \right) dx dy$$

$$B = \frac{Y h^3}{24(1-\sigma^2)}, \quad Y_{2D} = \frac{Y h}{1-\sigma^2}$$

$$\sigma_{2D} = \frac{\sigma}{1-\sigma}$$

A Landau like approach yield the same
with the generalization

$$\mathcal{F}_{\text{bend}} = B \int H^2 dx dy$$

↳ mean curvature.

Gauss Bonnet Theorem

$$\int \cancel{K} ds = \text{only depends on topology}$$