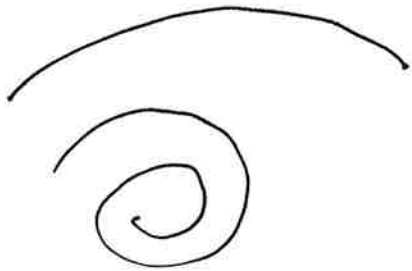


Filaments / slender rods

Nov 21

①



Filaments can be distorted into quite complex shapes.

They are general curves in 3D.

stretch



bend



twist



Each of these can contribute to ~~free~~ free energy. They can also happen together.

Stretch is simple; we will ignore it.

Twist is tricky, we will not consider it.

We are going to consider BENDING

Bending of rods

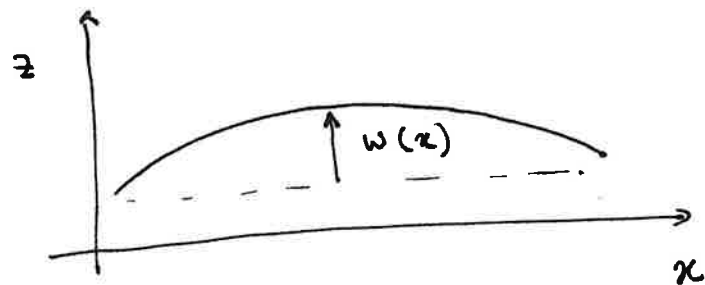
consider plane curves

$$\mathcal{J}_{\text{rod}}^{\text{bend}} = \frac{\kappa}{2} \int \left(\frac{d^2 w}{dx^2} \right)^2 dx$$

2D bending modulus

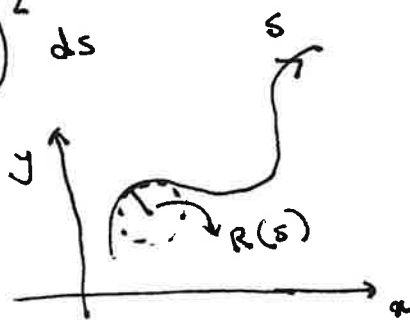
B has dimension
of energy

$[\kappa]$: energy · length.



$$= \frac{\kappa}{2} \int \left(\frac{1}{R} \right)^2 ds$$

coordinate independent
form



Large bending can happen without stretching.

Cantilever

Approximation:
a circle

Geometry:

Physics:

$\mathcal{J}$

$$L = R\theta, \quad d = R(1 - \cos\theta)$$

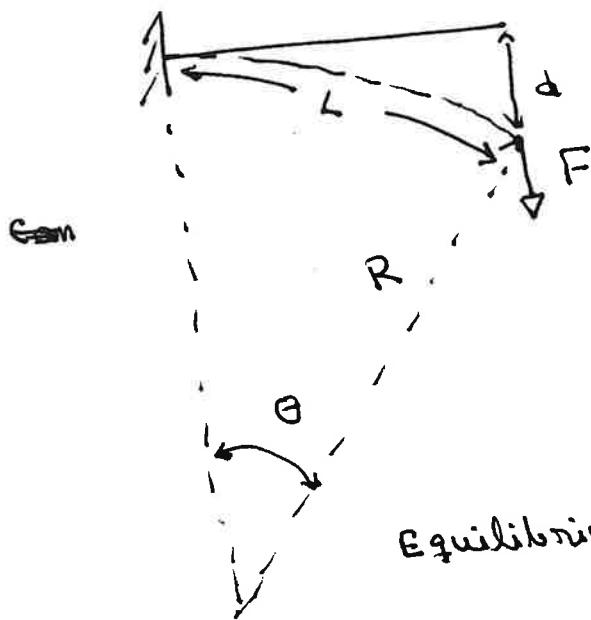
$$\mathcal{J} = \frac{\kappa L}{2R^2} - Fd = \frac{\kappa L}{2L^2} \theta^2 - \frac{FL}{\theta} (1 - \cos\theta)$$

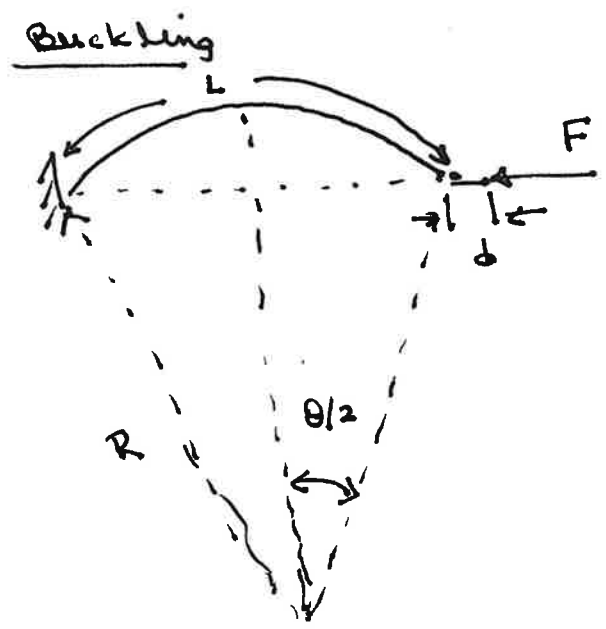
$$\approx \left(\frac{\kappa}{2L} \right) \theta^2 - \frac{FL\theta}{2}$$

$$\frac{\kappa}{2L} \cdot 2\theta - \frac{FL}{2} = 0 \Rightarrow \theta = \frac{FL^2}{2\kappa}$$

Equilibrium: $\frac{\partial \mathcal{J}}{\partial \theta} = 0$

$$\Rightarrow d \approx \frac{L}{\theta} \left(\frac{\theta^2}{2} \right) \sim \frac{FL^3}{4\kappa} \Rightarrow \boxed{d \propto F}$$





Approximation: Arc of a circle.

Geometry: $d = L - 2R \sin\left(\frac{\theta}{2}\right) ; L = R\theta$

$$= L - 2\frac{L}{\theta} \sin\left(\frac{\theta}{2}\right)$$

$$= L - \frac{2L}{\theta} \left[\frac{\theta}{2} - \frac{\theta^3}{48} + \frac{\theta^5}{3840} - \dots \right]$$

$$= L \left[1 - 1 + \frac{\theta^2}{3!} - \frac{\theta^4}{5!} + \dots \right]$$

Physics

$$U = \frac{kL}{2R^2} - Fd$$

$$\approx \frac{k\theta^2}{2L} - FL \left(\frac{\theta^2}{3!} - \frac{\theta^4}{5!} \right)$$

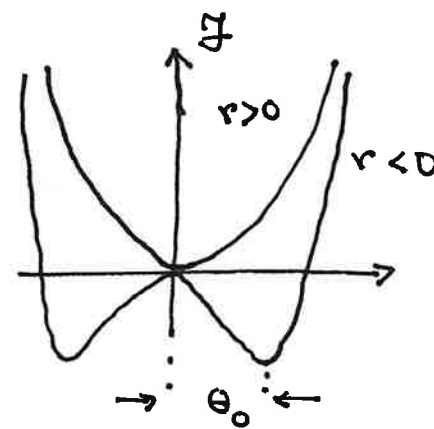
$$= \left(\frac{k}{2L} - \frac{FL}{3!} \right) \theta^2 + \frac{FL}{5!} \theta^4 = \frac{k}{2} \theta^2 + \frac{4}{5} \theta^4$$

$$r = 2 \left(\frac{k}{2L} - \frac{FL}{6} \right) = \left(\frac{k}{L} - \frac{FL}{3} \right)$$

$$u = \frac{4 \cdot FL}{5!} = \frac{FL}{30}$$

stability

$$\mathcal{J}(\theta) = \frac{r}{2} \theta^2 + \frac{u}{4} \theta^4$$



θ^4 model

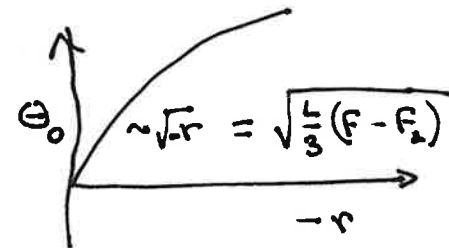
- For $r > 0$ there is only one minima at $\theta = 0$

- For $r < 0$ symmetry is spontaneously broken $\theta = \pm \theta_0$

Actually there are infinite number of configurations all with equal energy.

Critical buckling force: F_* , for $F = F_*$, $r = 0 \Rightarrow \boxed{F_* = \frac{3e}{L^2}}$

Equilibrium: $\frac{\partial \mathcal{J}}{\partial \theta} = 0 \Rightarrow r\theta + u\theta^3 = 0$
 $\Rightarrow \theta = 0$, or $\theta = \pm \sqrt{\frac{-r}{u}}$

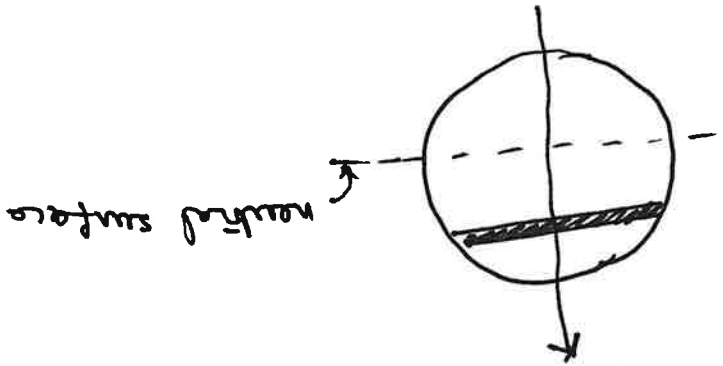


stability

$$\frac{\partial^2 \mathcal{J}}{\partial \theta^2} = r + 3u\theta^2$$

	r	or	$-2r$	
	exists, stable		does not exist	$r > 0$
	unstable		stable	$r < 0$

3

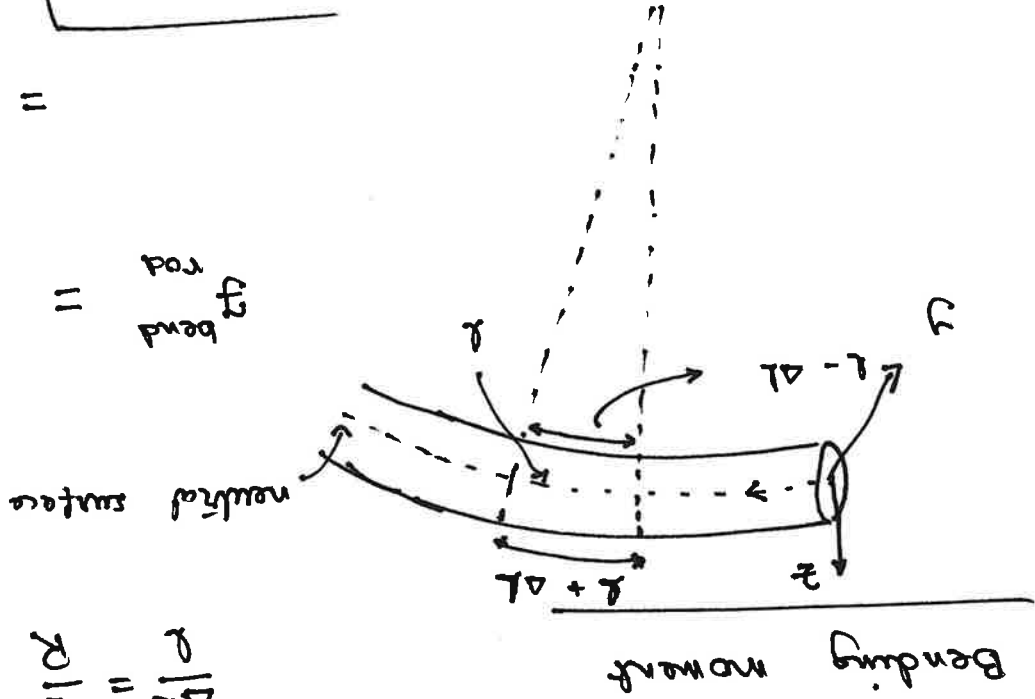


$$J_{\text{rod}}^{\text{bend}} = \gamma \int (\text{strain})^2 dx dy dz = \gamma \int \frac{dx}{R^2} \int z^2 dy dz$$

$$= \gamma I \int \frac{dx}{R^2} \quad ; \quad I = \int z^2 dy dz$$

one component of
the moment of
inertia tensor

$$\frac{\Delta l}{l} = \frac{z}{R}$$



$$\Rightarrow \boxed{\epsilon = \gamma I}$$

Summary (Filaments)

stretching: easy, we do not consider
 twist: tricky we do not consider
 bend: we do consider.

$$\mathcal{F}_{\text{rod}} = \frac{\kappa}{2} \int \left(\frac{1}{R} \right)^2 ds$$

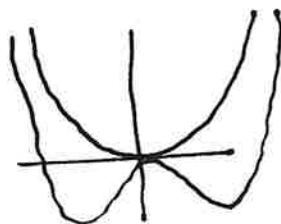
$R(s)$ is the radius of curvature.

To solve problems:

consider the ^{deformed} rod as an arc of a circle.
 calculate free energy. minimize it.

Bucking:

$$\mathcal{F} = \frac{r}{2} \Theta^2 + \frac{u}{4} \Theta^4$$



Is a typical problem

where spontaneous symmetry breaking is found.

A typical example of how an instability appears and to destabilize a solution that possesses symmetry and a new essentially nonlinear stable solution with broken symmetry appears.