

(1)

$$\nabla^2 \vec{v} = -\vec{\nabla} p \quad (1)$$

$$\vec{\nabla} \cdot \vec{v} = 0 \quad (2)$$

$$\vec{v} = (u, v) = (u(x, y), v(x, y))$$

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}; \quad \psi: \text{streamfunction.}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{v} = -\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

$$\omega = \vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & 0 \\ u & v & 0 \end{vmatrix} = \hat{z} (\partial_x v - \partial_y u) = \hat{z} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = \hat{z} \nabla^2 \psi$$

From (1): $\nabla^2 (\vec{\nabla} \times \vec{v}) = 0 \Rightarrow \nabla^2 \vec{\omega} = 0 \Rightarrow \boxed{\nabla^4 \psi = 0}$
Again the bi-harmonic eqn.

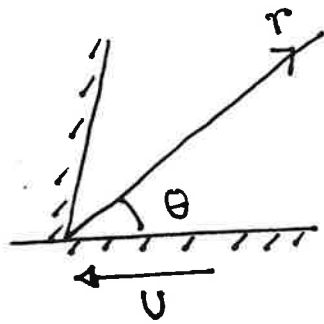
Define $\vec{\chi} = \frac{1}{2} \psi$, & $\vec{v} = \vec{\nabla} \times \vec{\chi}$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & 0 \\ 0 & 0 & \cancel{\psi} \end{vmatrix} = (\hat{x} \partial_y - \hat{y} \partial_x) \cancel{\psi}$$

Dimensionally: $\psi \approx \text{velocity} \cdot \text{length}.$

Stokes flow near corner

$$Re = \frac{rU}{\nu} \ll 1$$



Taylor scraper

why?

$$\psi = r^\lambda f(\theta)$$

substitute to $\nabla^4 \psi = 0$

to obtain: $\lambda^2(\lambda-2)^2 f(\theta) + [\lambda^2 + (\lambda-2)^2] f'' + f^{(iv)} = 0$ for $0 \leq \theta \leq \alpha$

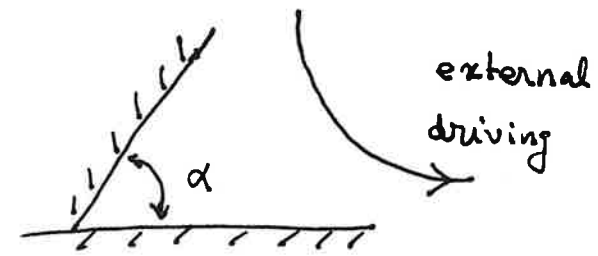
Taylor scraper: B.C.

$$\left. \begin{aligned} v_r &= -U & \text{at } \theta = 0 \\ v_\theta &= 0 & \text{at } \theta = 0 \end{aligned} \right\}$$

$$\left. \begin{aligned} v_r &= 0 \\ v_\theta &= 0 \end{aligned} \right\} \text{ at } \theta = \alpha$$

Dimensionally

$$\psi = Ur g(\theta) \Rightarrow \lambda = 1$$



Moffatt vortices

1. $\psi = R(r)f(\theta)$ is just assumption of separation of variable

2. $R(r) = r^\lambda$ because there is no length-scale in the problem.

$$\Rightarrow f(\theta) + 2f'' + f^{(iv)} = 0$$

$$\Rightarrow f(\theta) = A \cos \theta + B \sin \theta + C \theta \cos \theta + D \theta \sin \theta$$

Impose boundary condition to obtain

$$\psi = \frac{rU}{\alpha^2 - \sin^2 \alpha} \left[-\alpha^2 \sin \theta + \sin^2 \alpha \theta \cos \theta + (\alpha - \cos \alpha \sin \alpha) \theta \sin \theta \right]$$

Eventually you can calculate the stress:

$$\sigma_{\theta\theta} = -\frac{2\eta U}{r} \frac{\sin^2 \alpha}{\alpha^2 - \sin^2 \alpha}, \quad \sigma_{r\theta} = \frac{2\eta U}{r} \frac{\alpha - \sin \alpha \cos \alpha}{\alpha^2 - \sin^2 \alpha}$$

Goes to ∞ as $r \rightarrow 0$!!

How to get θ & λ from ψ ?

$$\frac{\lambda e}{\hbar e} \psi - \frac{\theta e}{\hbar e} \frac{\lambda}{r} \psi = \begin{vmatrix} \psi & 0 & 0 \\ r e & \theta e \frac{\lambda}{r} & \lambda e \\ r & \psi & \frac{\lambda}{r} \end{vmatrix} = \chi \times \Delta = \psi$$

$$0 = \theta \quad \text{for} \quad \begin{cases} n- = \frac{\theta e}{\hbar e} \frac{\lambda}{r} \\ 0 = \frac{\lambda e}{\hbar e} \end{cases} \quad \Leftarrow$$

$$\begin{cases} n- = (0) \frac{\lambda}{r} \\ 0 = (0) \frac{\lambda}{r} \end{cases} \quad \Leftarrow$$

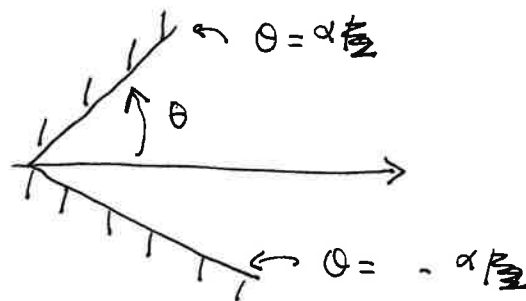
$$\lambda = \theta \quad \text{for} \quad \begin{cases} 0 = (x) \frac{\lambda}{r} \\ 0 = (x) \frac{\lambda}{r} \end{cases}$$

Moffat vortices

B.C.

(5)

$$\psi = \int_{\lambda} r^{\lambda} f_{\lambda}(\theta)$$



$$\begin{cases} f(-\alpha) = 0 \\ f'(-\alpha) = 0 \end{cases} \quad \begin{cases} f(\alpha) = 0 \\ f'(\alpha) = 0 \end{cases}$$

$$f_{\lambda}(\theta) = A \cos \lambda \theta + B \sin \lambda \theta + C \cos (\lambda - 2) \theta + D \sin (\lambda - 2) \theta$$

$$\lambda \neq 0, 1, 2$$

Assume anti-symmetry :

$$B = D = 0$$

$$A \cos \lambda \alpha + C \cos (\lambda - 2) \alpha = 0$$

$$A \lambda \sin \lambda \alpha + C (\lambda - 2) \sin (\lambda - 2) \alpha = 0$$

$$f(-\theta) = -f(\theta)$$

For a solution for A, C to exist

$$\begin{vmatrix} \cos \lambda \alpha & \cos (\lambda - 2) \alpha \\ \lambda \sin \lambda \alpha & (\lambda - 2) \sin (\lambda - 2) \alpha \end{vmatrix} = 0$$

=>

$$\boxed{\alpha \sin 2(\lambda - 1) \alpha = -(\lambda - 1) \alpha \sin 2 \alpha}$$

An example of solvability condition.

* Statements : (1) For $\alpha > 73^\circ$, λ has real solutions.

Then There are ~~infinite~~ ^{many} ~~number~~ solutions, in principle, but the smallest λ dominates at small r .

$$\psi \approx A r^{\lambda_1} \left[\cos \lambda_1 \theta \cos(\lambda_1 - 2) \alpha - \cos(\lambda_1 - 2) \theta \cos \lambda_1 \alpha \right]$$

(2) For $\alpha < \alpha_c \approx 73^\circ$ there are no real solutions.

$$\psi = \underset{\substack{\uparrow \\ \text{Real part}}}{r^{\lambda}} f(\theta) = r \cdot \underset{\substack{\uparrow \\ \text{can also} \\ \text{be complex}}}{r^p} \cdot \underset{\substack{\uparrow \\ g = \ln r}}{r^{iq}} f(\theta) = r^{p+1} \left[\cos(g\theta) + i \sin(g\theta) \right] f(\theta)$$

$$\lambda = 1 + p + iq, \quad \psi = r^{p+1} \operatorname{Re} \left[\left\{ \cos(g\theta) + i \sin(g\theta) \right\} f(\theta) \right]$$

From solvability condition:

$$\sin 2p\alpha \cosh 2q\alpha = -p \sin 2\alpha$$

$$\cos 2p\alpha \sinh 2q\alpha = -q \sin 2\alpha$$

$$* \operatorname{Re}[\psi] \approx r^{1+p} \left[\cos(q, \theta) F_1(\theta) + \sin(q, \theta) G_1(\theta) \right]$$

for one of the solutions.

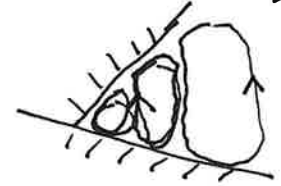
$$\alpha_1 = 1 + p_1 + iq_1$$

(8)

$$\Rightarrow \psi \left(e^{-2n\pi/q_1} r_0, \theta \right) = e^{-2n\pi(1+p_1)/q_1} \psi(r_0, \theta)$$

ψ is periodic structure (amplitude changes but the structure is periodic)

This gives an infinite set of counterrotating eddies



Note:

1. Vortical solutions of Stokes eqn. (Uncommon but not surprising on hindsight)

2. What happens to our dimensional argument?

3. Self-similarity of first & second kind.