

1

We start with the Navier-Stokes Equation

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} = \nu \nabla^2 \vec{v} - \nabla p / \rho + \text{Boundary conditions}$$

Let us try to see how small or large typical terms are.

$$\vec{v} \sim U, \quad x \sim L \Rightarrow t \sim \frac{L}{U}$$

$$\partial_t \vec{v} \sim \frac{U}{L}, \quad (\vec{v} \cdot \nabla) \vec{v} \sim \frac{U^2}{L}, \quad \nu \nabla^2 \vec{v} \sim \frac{\nu U}{L^2}$$

$$\frac{(\vec{v} \cdot \nabla) \vec{v}}{\nu \nabla^2 \vec{v}} \sim \frac{U^2 L^2}{L \nu U} \sim \frac{LU}{\nu} \equiv Re$$

$$\Rightarrow \text{As } Re \rightarrow 0 \quad \nu \nabla^2 \vec{v} \gg (\vec{v} \cdot \nabla) \vec{v} \quad (\text{Not always})$$

$$\text{and} \quad (\vec{v} \cdot \nabla) \vec{v} \sim \partial_t \vec{v}$$

Stokes Problem

$$\boxed{0 = \nu \nabla^2 \vec{v} - (\vec{v} \cdot \nabla) \vec{v} / \rho}$$

$\Rightarrow$

(2)

Stokes Problem:

$$\mu = 3\nu$$

$$\begin{aligned} \mu \nabla^2 \vec{v} &= \vec{\nabla} p \\ \vec{\nabla} \cdot \vec{v} &= 0 \end{aligned}$$

- No time
- Linear

Land of No Inertia

Creeping Flows

• Many varieties of Stokes Problems

A. Flows in narrow channels

examples

microfluidic, Tribology, Lubrication
Porous rocks,

B. Sedimenting bodies: or bodies moving with constant velocity.

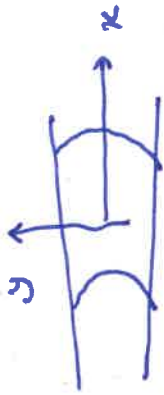


C Swimming microorganisms

How to solve Stokes Problems ?

3

A. Narrow channels : Lubrication approximation



Locally parabolic

- $\partial_x u + \partial_y v = 0$
- $\mu (\partial_x^2 u + \partial_y^2 v) = \partial_x p$ with $\partial_y^2 \gg \partial_x^2$
- $p = p(x)$ not of y

Use symmetry to simplify the problem.



B.1

$$\text{In 2-d : } \mu \nabla^2 \vec{v} = \vec{\nabla} p \Rightarrow \nabla^2 \vec{v} = 0 \quad \vec{v} = \vec{\nabla} \psi$$

$$\text{In 2d : } \vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} \psi_y \\ -\psi_x \end{pmatrix}$$

(4)

$\Rightarrow \vec{\omega}$ is a scalar $\omega(x, y) = \partial_x v_y - \partial_y v_x$

Now define: A function $\psi(x, y)$ such that-

$$v_y = \partial_x \psi, \quad v_x = -\partial_y \psi$$

$$\Rightarrow \omega = \partial_x^2 \psi + \partial_y^2 \psi = \nabla^2 \psi$$

$$\vec{\nabla} \cdot \vec{v} = \partial_x v_x + \partial_y v_y = -\partial_{xy} \psi + \partial_{yx} \psi = 0$$

Also

$$\boxed{\nabla^4 \psi = 0}$$

$\Rightarrow$ The Stokes problem becomes

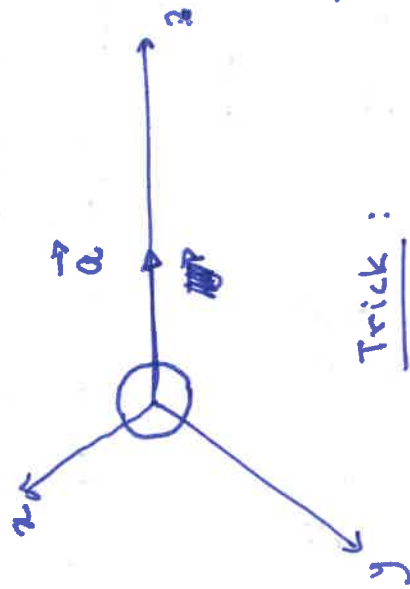
- Bi-Laplacian

with boundary condition on $\partial_x \psi$ and $\partial_y \psi$

B 2 / Solve the Stokes problem for a sphere moving with velocity $\vec{U}$ in an infinite space.

5

B2 Stokes Problem with axial symmetry



Solve for the flow with boundary condition $\vec{v} = 0$ on the surface of the sphere.

Trick: $\vec{v} = \text{some function of } (x, y, z, \vec{a})$

The function must have axial symmetry.

Ansatz

$$v_\alpha = a^\mu \psi_\mu(x, y, z)$$

fancy word
for a guess

→ some operator of rank 2.

$$\psi_\mu = \left(\delta_{\mu\nu} \frac{\partial^2}{\partial x_\mu \partial x_\nu} - \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \right)$$

$$p = \mu a_\beta \pi_\beta \quad - \text{I don't know } \pi_\beta$$

6

Why?

1. $\vec{v}$ must depend linearly on $\vec{a} \Rightarrow v_\alpha = a_\alpha \phi(x, y, z)$
2. Can the operator ψ have first order derivatives?

a_α ∂_β ψ

only way I can make a vector out of this is

$$v_\alpha = \epsilon_{\alpha\beta\gamma} a_\beta \partial_\gamma \psi = \vec{a} \times (\vec{\nabla} \psi)$$

$$\Rightarrow \begin{cases} \vec{a}' = -\vec{a}, \\ \vec{\nabla}' = -\vec{\nabla} \end{cases} \quad \text{and} \quad \vec{v}' \rightarrow -\vec{v}$$

$$\text{Let } \begin{cases} x \rightarrow -x, \\ y \rightarrow -y \\ z \rightarrow -z \end{cases}$$

But cross-products do not change sign under parity, so $\vec{v}$ would remain unchanged.

(7)

3. Then ψ can have 2nd order derivatives.

$$\Rightarrow U_{\alpha\beta} = \delta_{\alpha\beta} \frac{\partial^2}{\partial x_\mu \partial x_\mu} + \frac{\partial^2}{\partial x_\alpha \partial x_\beta}$$

↑ we choose - on hindsight

could there be different numbers in front of the two terms? NO. Because the asymmetry between the coordinates is given by $\vec{a}$ which is already taken into account.

4. Third order derivative? No same as 2.

5. 4th and other higher order even derivatives?

YES but they are more complicated.

If a simpler solution satisfies the equation then by uniqueness we do not have to look further.

8

Substitute in Stokes Eqn.

$$\mu \partial_{kk} v_\alpha = \partial_\alpha p \Rightarrow \mu \partial_{kk} a_\beta \left(\frac{\partial^2}{\partial x_\mu \partial x_\mu} - \frac{\partial^2}{\partial x_\alpha \partial x_\alpha} \right) \psi = \mu a_\beta \partial_\alpha \partial \Pi_\beta$$

$$\Rightarrow \partial_\mu^2 \partial_k^2 \psi - \partial_\alpha^2 (\partial_k^2 \partial_\beta \psi) = \partial_\alpha \Pi_\beta$$

$$\boxed{\nabla^4 \psi = 0}$$

$$\Pi_\beta = - \partial_\beta \partial_k^2 \psi$$

and

$$\boxed{\vec{\Pi} = - \vec{\nabla} (\nabla^2 \psi)}$$

or

the streamfunction.

This ψ is NOT the second derivative of ψ

Remember:

$$v_\alpha = 10 a_\alpha - \left(\frac{3}{4} a_\beta \frac{x_\alpha x_\beta}{r^3} + \frac{a_\alpha}{r} \right) - \frac{1}{4} \frac{\partial}{\partial x_\alpha} \left(\frac{v_\beta x_\beta}{r^3} \right)$$

9

Green's function :

At large r , $v \sim \frac{1}{r}$
Notice
 what is the solution of $\nabla^2 v - \nabla p = \vec{a} \delta(\vec{r})$

This has the same axial symmetry; and $v \sim \frac{1}{r}$ for larger r

$$v = \frac{r}{8\pi}$$

solution

~~moment~~

Ans

~~we~~ Substitute in the (inhomogeneous) Stokes Eqn.

$$\nabla^4 \psi = \delta(\vec{r})$$

$$\nabla^2 \Phi = \delta(\vec{r})$$

or

$$\text{with } \Phi = \nabla^2 \psi$$

Solution : $\Phi = \frac{1}{4\lambda r}$

$$\Rightarrow \nabla^2 \psi = \frac{1}{4\lambda r}$$

$$\Rightarrow \frac{1}{r} \frac{\partial^2}{\partial r^2} (r\psi) = \frac{1}{4\lambda r}$$

$$\Rightarrow \frac{\partial}{\partial r} (r\psi) = \frac{r}{4\lambda} + A$$

$$\Rightarrow r\psi = \frac{r^2}{8\lambda} + Ar + B$$

$$\Rightarrow \psi = \frac{r}{8\lambda} + A + \frac{B}{r}$$

By boundary condition $B, A = 0$

$$u_\alpha = \frac{1}{8\lambda} \left(\frac{x_\beta}{r} + \frac{x_\alpha x_\beta}{r^3} \right) a_\beta$$

$$p = \frac{1}{4\lambda} \left(\frac{x_\beta}{r^3} \right) a_\beta$$

Stokeslet solution

Green's function for Stokes Eqn