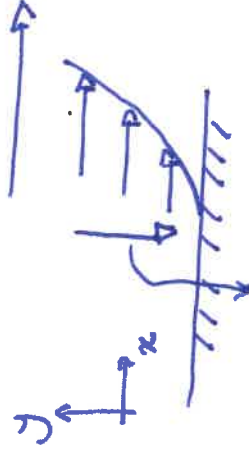


Equations for fluids

$$\partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla \frac{p}{\rho} - \nabla \psi + \left(\text{something else} \right)$$

↑
must come
from off-diagonal
component of stress

Experimental fact : Flow velocity (all components) go to zero
at a solid surface.



x component of momentum
getting transported in the y
direction. An off-diagonal
component of stress.

~~Fluids~~

$$\sigma_{\alpha\beta} = \left(\uparrow \right) \frac{\partial v_\mu}{\partial x_\mu}$$

A 4th rank tensor
that is a property
of fluids

Symmetry & conservation for fluids :

(2)

Any conserved density ρ has a current $\vec{J}$

$$\boxed{\partial_t \rho + \text{div } \vec{J} = 0}$$

such that

If ρ is mass density $\vec{J} = \rho \vec{v}$

$\Rightarrow \partial_t \rho + \text{div}(\rho \vec{v}) = 0$ — continuity eqn.

Let ρ be momentum density : $\pi_\alpha \equiv \rho v_\alpha$

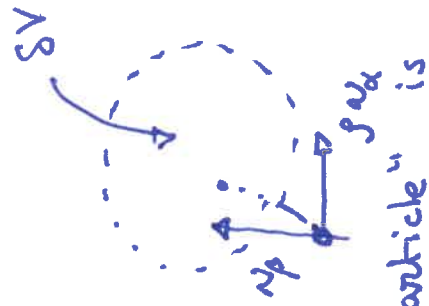
$$J_{\text{momentum}} = (\rho v_\alpha) v_\beta$$

But momentum can ~~also~~ also change

due to external (external to volume δV)

forces :

$$\Rightarrow \partial_t (\rho v_\alpha) + \text{div}(\rho v_\alpha v_\beta) = (F/\delta V) = \partial_\beta \sigma_{\alpha\beta}$$



A "particle" is momentum ρv_α being transported by flow v_β

$$\Rightarrow \partial_t (S v_\alpha) + \text{div} (S v_\alpha v_\beta + \sigma_{\alpha\beta}) = 0$$

(3)



$-p \delta_{\alpha\beta}$
fluid at rest

-3Ψ
gravitational potential

$\sigma_{\alpha\beta}^v \rightarrow$ something else that depends on fluid in motion

$\sigma_{\alpha\beta}^v \rightarrow$ must depend on velocity field $\vec{v}(x, t)$

But cannot depend on $\vec{v}$ itself if it did we could set it to zero by a Galilean transformation.

Demand: Our theory must have

Galilean invariance.

consequence:

$$\sigma_{\alpha\beta}^v = \left(\text{wavy line} \right) \frac{\partial v_\alpha}{\partial x_\beta} + \left(\text{wavy line} \right) \frac{\partial v_\beta}{\partial x_\alpha}$$

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consequence:
$$\sigma_{\alpha\beta}^v = \left(\right) \frac{\partial v_\mu}{\partial x_\nu} + \left(\right) \frac{\partial v_\mu}{\partial x_\nu} \frac{\partial v_\sigma}{\partial x_\mu} + \dots$$

— A Gradient expansion

If we truncate it to lowest order

$$\sigma_{\alpha\beta}^v = \eta \frac{\partial v_\mu}{\partial x_\nu} \quad \begin{matrix} \nearrow \text{rate of rotation} \\ \searrow \text{rate of strain} \end{matrix} \quad \vec{\omega} = \vec{\nabla} \times \vec{v} \quad S_{\alpha\beta} = \frac{1}{2} \left(\frac{\partial v_\beta}{\partial x_\alpha} + \frac{\partial v_\alpha}{\partial x_\beta} \right)$$

Because rotation cannot be responsible for generating a force.



4th rank isotropic tensor

Force $\frac{\partial v_\mu}{\partial x_\nu} + \frac{\partial v_\nu}{\partial x_\mu}$

$$\sigma_{\alpha\beta}^v = \mu \left(S_{\alpha\beta} - \frac{2}{3} \delta_{\alpha\beta} S_{\kappa\kappa} \right) + \int \delta_{\alpha\beta} S_{\kappa\kappa}$$

⑤

$$S_{\alpha\beta} = \left(\partial_\alpha v_\beta + \partial_\beta v_\alpha \right)$$

μ : shear viscosity

$\int$: bulk viscosity

A minor difference
from elasticity

$$\sigma_{\alpha\beta} = K \delta_{\alpha\beta} S_{\kappa\kappa} + 2\mu \left(S_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} S_{\kappa\kappa} \right)$$

For incompressible fluids: $S_{\kappa\kappa} = 0$

$$\sigma_{\alpha\beta}^v = \mu \left(\partial_\alpha v_\beta + \partial_\beta v_\alpha \right)$$

$$\Rightarrow \partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = - \frac{\vec{\nabla} p}{\rho} - \vec{\nabla} \psi$$

$$- \partial_\beta \sigma_{\alpha\beta}^v$$

incompressibility

Navier-Stokes
Equation

$$\Rightarrow \left[\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = - \frac{\vec{\nabla} p}{\rho} - \vec{\nabla} \psi + \frac{\mu}{\rho} \vec{\nabla}^2 \vec{v} \right]$$

Revisit

Momentum conservation $\leftrightarrow$

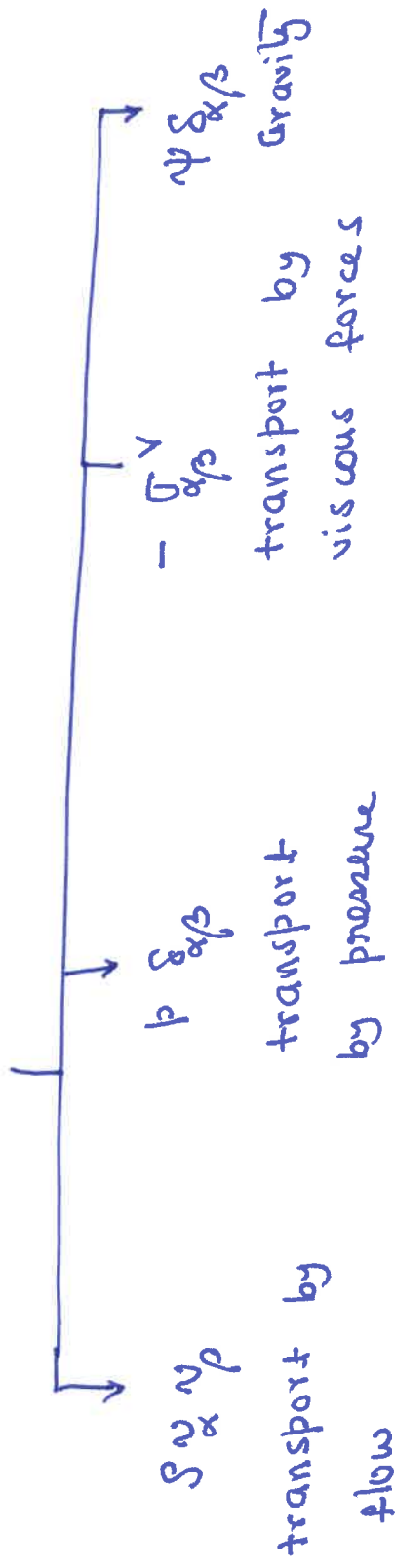
Symmetry under spatial translation

$$\partial_t \pi_\alpha + \text{div} (P_{\alpha\beta}) = 0$$

momentum density

momentum current

$$= S v_\alpha$$



$$\sigma_{\alpha\beta}^v = \eta_{\alpha\beta\mu\nu} (\partial_\mu v_\nu + \partial_\nu v_\mu) \rightarrow \text{Galilean invariance}$$

$$= \mu \left(S_{\alpha\beta} - \frac{2}{3} \delta_{\alpha\beta} S_{kk} \right) + \int \delta_{\alpha\beta} S_{kk} \leftrightarrow \text{symmetry under rotation}$$

$$= \mu S_{\alpha\beta} \leftrightarrow \text{incompressibility}$$

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{\nabla p}{\rho} - \nabla \psi + \nu \nabla^2 \vec{v}$$

$$\nabla \cdot \vec{v} = 0$$

$$\nu = \frac{\mu}{\rho}$$

μ : dynamic viscosity

ν : kinematic viscosity

The Incompressible Navier-Stokes Eqn.

Note 1. The equation of state is irrelevant.

$$\text{Take divergence} \quad \nabla \cdot [(\vec{v} \cdot \nabla) \vec{v}] = \nabla^2 p \quad \left| \begin{array}{l} \rho = 1 \\ \psi = 0 \end{array} \right.$$

$\Rightarrow$ p is determined by solving a Laplace eqn. at every instant of time

$$\text{Note 2} \quad \partial_t \vec{v} + (\vec{v} \times \vec{v}) = -\nabla \left(\frac{p}{\rho} + \psi + \frac{v^2}{2} \right) + \nu \nabla^2 \vec{v}$$

$$\text{Take a curl:} \quad \vec{\omega} = \nabla \times \vec{v}$$

$$\partial_t \vec{\omega} + \nabla \times (\vec{v} \times \vec{v}) = \nu \nabla^2 \vec{\omega}$$

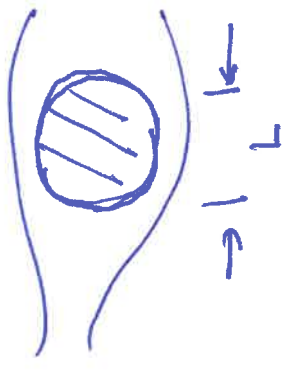
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U : typical velocity scale

L : typical length scale

$$\Rightarrow T \equiv \frac{L}{U} = \text{typical time scale}$$



Redefine

~~Figure~~

$$\begin{aligned} \vec{v} &= U \vec{v}_i \\ x &= L x_i \\ t &= T t_i \end{aligned}$$

$$\left. \begin{aligned} \vec{v}_i \\ x_i \\ t_i \end{aligned} \right\} \text{Are dimensionless}$$

$$\Rightarrow \frac{\partial}{\partial t} = \frac{\partial}{\partial t_i} \frac{\partial t_i}{\partial t} = \frac{1}{T} \frac{\partial}{\partial t_i}$$

$$\frac{\partial}{\partial x} = \frac{\partial x_i}{\partial x} \frac{\partial}{\partial x_i} = \frac{1}{L} \frac{\partial}{\partial x_i}$$

$$\vec{\nabla} = \frac{1}{L} \vec{\nabla}_i$$

$$\frac{\partial}{\partial t} \rightarrow \frac{1}{T} \cdot U \frac{\partial}{\partial t_i}$$

$\Rightarrow$

$$(\vec{v} \cdot \vec{\nabla}) \vec{v} \rightarrow \left(\vec{v}_i \cdot \frac{1}{L} \vec{\nabla}_i \right) U \vec{v}_i$$

9)

$$\hat{13} = \hat{\nabla} \times \hat{2} \quad \uparrow \quad \hat{13} = \hat{\nabla} \times \hat{2}$$

$$\Rightarrow \hat{13} \times \hat{2} \quad \uparrow \quad \hat{13} \times \hat{2}$$

Putting everything together

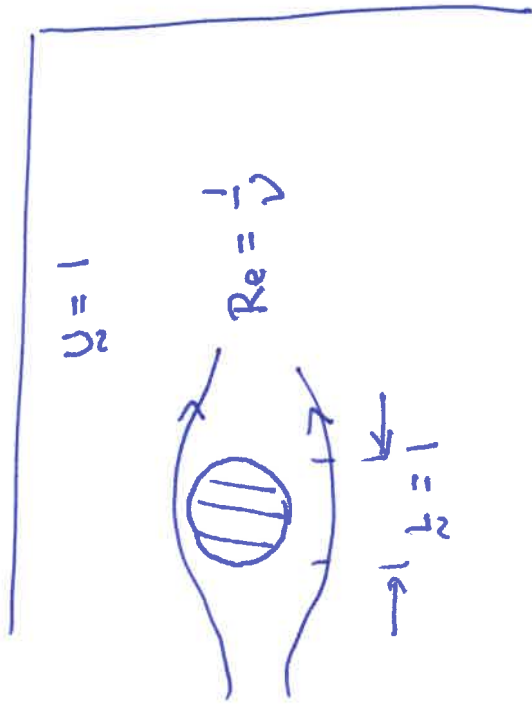
$$\frac{\partial \hat{13}}{\partial t} + \hat{13} \times \hat{2} = \hat{\nabla} \times \left(\hat{13} \times \hat{2} \right)$$

$$\Rightarrow \frac{\partial \hat{13}}{\partial t} + \hat{13} \times \left(\frac{\partial \hat{13}}{\partial t} \right) = \left(\frac{\partial \hat{13}}{\partial t} \right) \times \hat{2}$$

$$= \frac{1}{\text{Re}} \hat{\nabla}^2 \hat{13}$$



The only quantity that depends on the dimensional parameters of the problem



Dynamical similarity :

If we solve the problem for one Re then if we change L , U and ν in such a way that Re remains the same, the solution also remains the same.

$$\left[\begin{array}{l} \partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} = \frac{1}{\text{Re}} \nabla^2 \vec{v} - \nabla p \\ \nabla \cdot \vec{v} = 0 \end{array} \right]$$

Non-dimensional.

Remember :

$$\vec{v} = U \vec{v}_2$$

$$\nabla = \frac{1}{L} \nabla_2$$

$$p/\rho = \left(\frac{p}{\rho} \right)_2 (\#)$$

$$\nabla^2 (p/\rho) = \nabla \cdot [(\vec{v} \cdot \nabla) \vec{v}]$$

$$\Rightarrow \frac{1}{L^2} \nabla_2^2 \left(\frac{p}{\rho} \right) (\#) = \frac{U^2}{L^2} \nabla_2 \cdot [(\vec{v}_2 \cdot \nabla_2) \vec{v}_2]$$

$$\Rightarrow (\#) = U^2$$

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In the limit $Re \rightarrow \infty, \nu \rightarrow 0 \rightarrow \text{Difficult}$
 $Re \rightarrow 0, \nu \rightarrow \infty \rightarrow \text{Easy}$

As $Re \rightarrow 0$: $\nabla^2 \vec{u} + Re (\quad) = 0$

$\Rightarrow \boxed{\nabla^2 \vec{u} = 0}$ Stokes Eqn.

As $Re \rightarrow \infty$: $\partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla p$ Euler eqn.
 WRONG