

Navier Cauchy Equation

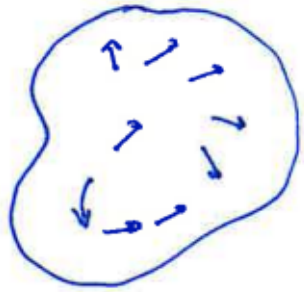
(1)

what we know

For a soft body

Deformation field: $\vec{u}(\vec{x})$

Gradient $G_{\alpha\beta} = \frac{\partial u_\alpha}{\partial x_\beta}$



second rank tensor

Strain (symmetrized gradient)

$$= \frac{1}{2} \left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right)$$

Stress: force per unit area $\sigma_{\alpha\beta}$

Eqn. of mechanical equilibrium: $\partial_\beta \sigma_{\alpha\beta} = 0$ [For no ext. forces]

with

$\sigma_{\alpha\beta} n_\beta = 0$ at the boundary

Free energy of deformation

$$F = E_{\alpha\beta\mu\nu} S_{\alpha\beta} S_{\mu\nu} \quad (\text{implicit assumption: small strain})$$

$$K = \lambda + \frac{2}{3} \mu$$

$\uparrow$ Bulk modulus
 $\uparrow$ Lamé coefficients
 $\leftarrow$ shear modulus

$$\begin{aligned}
 &= \frac{1}{2} K S_{\beta\beta}^2 + \mu \left(S_{\alpha\gamma} - \frac{1}{3} \delta_{\alpha\gamma} S_{\beta\beta} \right)^2 \\
 &= \frac{1}{2} \lambda S_{\beta\beta}^2 + \mu S_{\alpha\gamma}^2
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} &= \frac{1}{2} K S_{\beta\beta}^2 + \mu \left(S_{\alpha\gamma} - \frac{1}{3} \delta_{\alpha\gamma} S_{\beta\beta} \right)^2 \\ &= \frac{1}{2} \lambda S_{\beta\beta}^2 + \mu S_{\alpha\gamma}^2 \end{aligned}} \right\} \text{Isotropic material}$$

Now we have all the ingredients to write the equation of motion of a soft body:

$$\rho \frac{d^2 u_\alpha}{dt^2} = \partial_\beta \tau_{\alpha\beta}$$

$$= \partial_\beta \bar{E}_{\alpha\beta\mu\nu} S_{\mu\nu}$$

$$\Rightarrow \left\{ \rho \frac{d^2 u_\alpha}{dt^2} = \bar{E}_{\alpha\beta\mu\nu} \partial_\beta \left[\frac{1}{2} \left(\frac{\partial u_\mu}{\partial x_\nu} + \frac{\partial u_\nu}{\partial x_\mu} \right) \right] \right\}$$

$\uparrow$
 2nd order
 in time

$\uparrow$
 Assumed to
 be constant

$\uparrow$
 2nd order
 in space

Correct for any homogeneous
 soft body (Isotropic assumption
 not used so far)

with
 Initial
 &

$$\vec{u}(\vec{x}, 0)$$

Boundary
 conditions

we shall revisit
 this aspect.

Note:

1. Linear equation:

- (a) Principle of superposition works
- (b) Symmetry of boundary condition is reflected in the solutions
- (c) Solution is unique
 (once initial and boundary conditions are specified)

Isotropic soft body in static mech. equilibrium

← Actually 3 equations: ($\alpha = 1, 2, 3$)

$$\partial_\beta \nabla_{\alpha\beta} = 0$$

$$\Rightarrow \partial_\beta \left[K \delta_{\alpha\beta} s_{\mu\mu} + 2\mu \left(s_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} s_{\mu\mu} \right) \right] = 0$$

$$\partial_\beta \delta_{\alpha\beta} s_{\mu\mu} = \partial_\alpha s_{\mu\mu} = \frac{\partial}{\partial x_\alpha} \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) = \vec{\nabla} (\text{div } \vec{u})$$

$$\partial_\beta s_{\alpha\beta} = \frac{\partial}{\partial x_\beta} \left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} \right) \rightarrow \frac{\partial}{\partial x_\beta} \frac{\partial u_\beta}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha} \frac{\partial u_\beta}{\partial x_\beta} = \vec{\nabla} (\text{div } \vec{u})$$

$$\frac{\partial^2 u_\alpha}{\partial x_\beta \partial x_\beta}$$

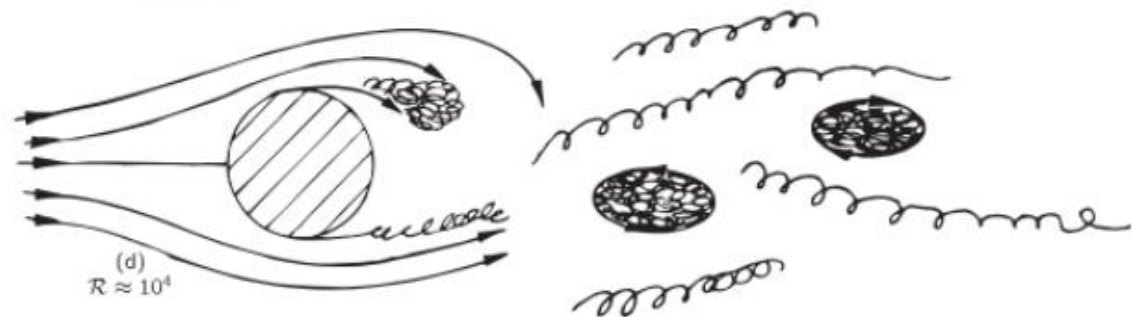
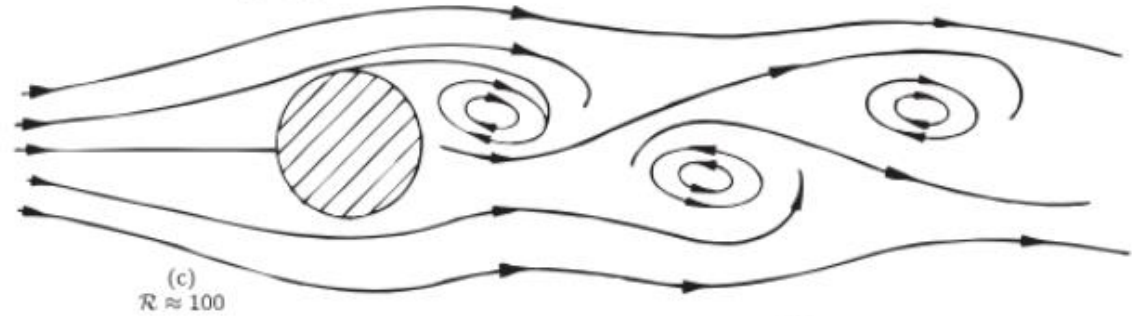
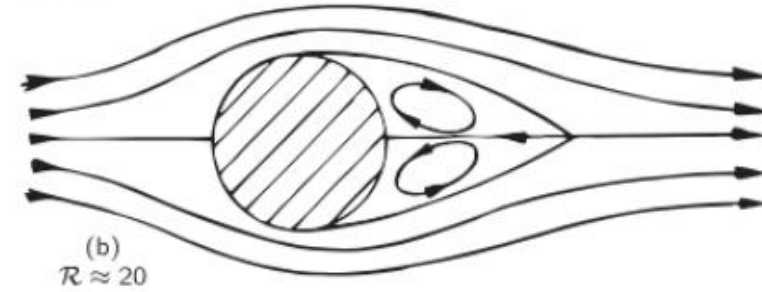
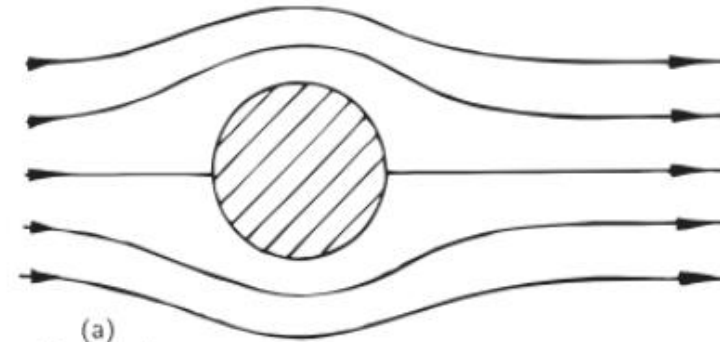
$$\nabla^2 \vec{u}$$

After rearranging

$$\boxed{(\lambda + \mu) \vec{\nabla} (\text{div } \vec{u}) + \mu \nabla^2 \vec{u} = 0}$$

Navier - Cauchy equation.

- Uniqueness
- Superposition
- Symmetry



A different (actually the same) route of Navier Cauchy eqn.

$$\mathcal{F}[\vec{u}] = \int E_{\alpha\beta\mu\nu} s_{\alpha\beta} s_{\mu\nu} d^3x$$

$\vec{u}$ is determined by minimizing the free energy.

functional
(not a function)

Analogy from mechanics

$$\mathcal{L} = T - V = \frac{1}{2} m \dot{q}^2 - V(q)$$

$$\text{Action } \mathcal{A}[q] = \int_{t_1}^{t_2} \mathcal{L}(t) dt$$

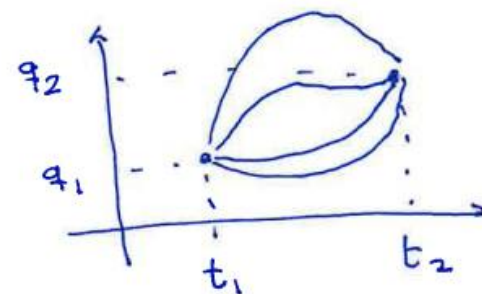
The path $q(t)$ actually followed by the particle is given by the principle of least action

$$\Rightarrow -m \frac{d}{dt}(\dot{q}) - \frac{\partial V}{\partial q} = 0$$

$$\Rightarrow \left| m \frac{d^2 q}{dt^2} = -\frac{\partial V}{\partial q} \right|$$

Newton's law

Equivalent



which path does the particle take?

~~Equation of~~ ~~not~~

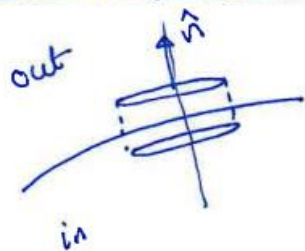
Navier - Cauchy eqn:

$$\frac{\delta \mathcal{F}}{\delta u} = 0$$

Questions :

- (a) Is $\mathcal{F}$ Helmholtz free energy or Gibbs free energy?
- (b) Are K and μ positive?

Boundary condition (very carefully)



A very short pillbox

Newton's law: $\sum \text{all forces} = 0$ (because its accⁿ is zero) (6)

$$\Rightarrow \vec{F}_{\text{int}} + \vec{F}_{\text{ext}} = 0$$

$$\Rightarrow \Delta V \partial_\beta \sigma_{\alpha\beta} + F_\alpha^{\text{ext}} = 0$$

$$\Rightarrow \partial_\beta \sigma_{\alpha\beta} + \frac{F_\alpha^{\text{ext}}}{\Delta V} = 0$$

$$\Rightarrow \partial_\beta \sigma_{\alpha\beta} + f_\alpha^{\text{ext}} = 0$$

Integrate this over the pillbox $\xrightarrow{\text{force per unit vol. (force density)}}$

Gauss's Theorem $\int_V \partial_\beta \sigma_{\alpha\beta} dV + \int_V f_\alpha^{\text{ext}} dV = 0$

$$\Rightarrow \oint_S \sigma_{\alpha\beta} n_\beta dS + \int_V P_\alpha \delta(z) dx dy dz = 0$$

$$\Rightarrow (\sigma_{\alpha\beta}^o - \sigma_{\alpha\beta}^i) n_\beta (dy dx) + P_\alpha (dy dx) = 0$$

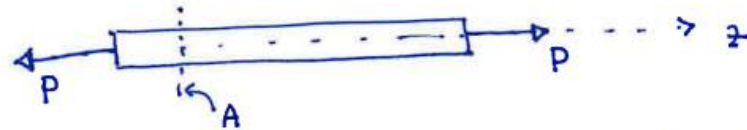
$$\Rightarrow \boxed{(\sigma_{\alpha\beta}^i - \sigma_{\alpha\beta}^o) n_\beta = P_\alpha}$$

$$f_\alpha^{\text{ext}} = P_\alpha \delta(z)$$

Dirac δ function

The force is a surface force.

Simplest application:



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*. A rod with ~~rectangular~~ square cross section. An external force F^{ext} is applied along the axis of the rod in opposite direction. The cross section has area ΔA . Then external stress $P = F^{ext}/\Delta A$

We take z axis along the axis of the rod.

1. Consider two parts of the rod separated at A. Clearly from free body diagram the force is constant ~~through the rod~~ along the rod. Hence internal stress σ_{zz} must also be constant. (Because ΔA does not change with z). As stress & strain are linearly related at every point ϵ_{zz} must also be constant.

2. At a side ($\hat{n} = \hat{x}$, or $\hat{n} = \hat{y}$):
 $\hat{n} = \hat{x} = (1, 0, 0)$
 $\sigma_{11}^i - \sigma_{11}^o = P_1 = 0$ (no force on the side)
 $\Rightarrow \sigma_{11}^i = 0$ ($\because \sigma_{11}^o = 0$)

The cross section is so small the σ_{11} is constant over the cross section. Hence $\sigma_{11} = 0$ everywhere.

3. Similarly $\hat{n} = \hat{x}$, $\sigma_{12} = 0$, $\hat{n} = \hat{y}$: $\sigma_{22} = 0$

(8)

4. At the boundary at the front:
 $\hat{n} = \hat{z}$, $\sigma_{\alpha 3} = P_3 = P \Rightarrow \sigma_{33} = P$, $\sigma_{23} = \sigma_{13} = 0$

5. so we have $\sigma_{\mu\mu} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{33} = P$

6. $s_{\alpha\beta} = \frac{1}{9K} \delta_{\alpha\beta} \sigma_{\mu\mu} + \frac{1}{2\mu} \left(\sigma_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} \sigma_{\mu\mu} \right)$

$\Rightarrow s_{zz} = \frac{1}{9K} \sigma_{zz} + \frac{1}{2\mu} \left(\sigma_{zz} - \frac{1}{3} \sigma_{zz} \right) = \left(\frac{1}{9K} + \frac{1}{3\mu} \right) P$

7. $s_{xx} = \frac{1}{9K} \sigma_{xx} + \frac{1}{2\mu} \left(\sigma_{xx} - \frac{1}{3} \sigma_{zz} \right) = \left(\frac{1}{9K} - \frac{1}{6\mu} \right) P$

From high school $\frac{P}{s_{zz}} = \frac{\text{stress}}{\text{strain}} = \text{Young's modulus} = Y$

$\Rightarrow \boxed{Y = \frac{9K\mu}{3K + \mu}}$

$\boxed{\sigma = \frac{1}{2} \frac{3K - 2\mu}{3K + 2\mu}}$

And $s_{xx} = -\sigma s_{zz}$
 $\downarrow$
 Poisson's ratio

Useful formulas

⑨ ⑩

$$\sigma_{\alpha\beta} = K \delta_{\alpha\beta} s_{\mu\mu} + 2\mu \left(s_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} s_{\mu\mu} \right)$$

↖ Invert ↙

$$s_{\alpha\beta} = \frac{1}{9K} \delta_{\alpha\beta} \sigma_{\mu\mu} + \frac{1}{2\mu} \left(\sigma_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} \sigma_{\mu\mu} \right)$$

$$F = \frac{Y}{2(1+\sigma)} \left(s_{\alpha\beta}^2 + \frac{\sigma}{1-2\sigma} s_{\mu\mu}^2 \right)$$

$$\sigma_{\alpha\beta} = \frac{Y}{1+\sigma} \left(s_{\alpha\beta} + \frac{\sigma}{1-2\sigma} \delta_{\alpha\beta} s_{\mu\mu} \right)$$

↖ Invert ↙

$$Y s_{\alpha\beta} = (1+\sigma) \sigma_{\alpha\beta} - \sigma \delta_{\alpha\beta} \sigma_{\mu\mu}$$