

Navier Cauchy Eqn. : solutions

$$(\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} = -\mathbf{f}_\alpha^{\text{ext}}$$

- Uniqueness
- Superposition
- Symmetry

If I can guess a solution it is the solution

1. Green's function:

Ancient history:

Linear operator $\rightarrow \mathcal{L} u = f$

↑
field

↖
source

$\mathcal{L} u = 0$ Homogeneous eqn.

$\neq f \leftarrow$ In homogeneous part

We seek a solution of the form:

$$u(x) = \int G(x-y) f(y) dy$$

$$\Rightarrow \mathcal{L} u = \int \mathcal{L}(x) G(x-y) f(y) dy = f(x)$$

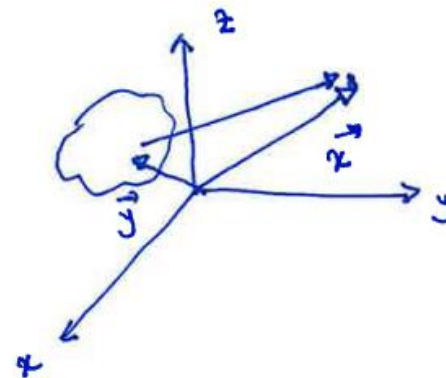
$$\Rightarrow \mathcal{L}(x) G(x-y) = \delta(x-y)$$

example:

$\nabla^2 \phi = \rho / \epsilon_0$ Poisson Eqn.

↖
charge density

solution: $\phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{y}) d\vec{y}}{|\vec{x} - \vec{y}|}$

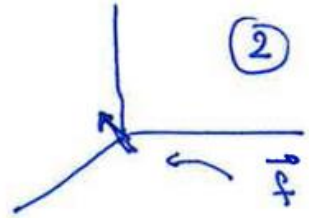


Navier - Cauchy Eqn :

$$\frac{1}{1-2\sigma} \vec{\nabla} (\text{div } \vec{u}) + \nabla^2 \vec{u} = - \frac{2(1+\sigma)}{Y} \vec{f} \delta(\vec{r})$$

(2)

volume density
of force



• Poisson Eqn. was a scalar eqn. This is a vector one.

we expect

$$u_{\alpha}(\vec{R}) = \int G_{\alpha\beta}(\vec{R}-\vec{r}) f_{\beta}(\vec{r}) d^d r$$

2nd rank tensor

Trick 1 :

$$\vec{u} = \vec{v} + \vec{w}$$

such that

$$\nabla^2 \vec{v} = - \frac{2(1+\sigma)}{Y} \vec{f} \delta(\vec{r})$$

This is 3 Poisson eqn. one for each component.

$$\Rightarrow \vec{v} = - \frac{2(1+\sigma)}{4\pi Y} \vec{f} \cdot \left(\frac{1}{r} \right)$$

Then
$$\frac{1}{1-2\sigma} \vec{\nabla} (\text{div}(\vec{v} + \vec{w})) + \nabla^2 \vec{w} = 0$$

$\vec{\nabla} \times$
$$0 + \nabla^2 (\vec{\nabla} \times \vec{w}) = 0 \Rightarrow \vec{\nabla} \times \vec{w} = 0 \Rightarrow \vec{w} = \vec{\nabla} \phi$$

substitute back in NC eqn.

$$\frac{1}{1-2\sigma} \vec{\nabla} \text{div}(\vec{v} + \vec{\nabla} \phi) + \nabla^2 (\vec{v} + \vec{w}) = - \frac{2(1+\sigma)}{Y} \vec{f} \delta(\vec{r})$$

↑ ↑ cancels

$$\Rightarrow (1-2\sigma)^{-1} \vec{\nabla} \text{div} \vec{v} + (1-2\sigma)^{-1} \underbrace{\vec{\nabla} (\nabla^2 \phi)}_{\uparrow \text{ same}} + \underbrace{\nabla^2 (\vec{\nabla} \phi)}_{\uparrow \text{ same}} = 0$$

$$\Rightarrow 2(1-\sigma) \nabla^2 (\vec{\nabla} \phi) = - \vec{\nabla} (\text{div} \vec{v})$$

← As we know $\vec{v}$ already we can calculate this

(4)

$$\Rightarrow 2(1-\sigma) \nabla^2 (\vec{\nabla} \phi) = \frac{1+\sigma}{2\pi Y} \vec{\nabla} (\vec{f} \cdot \vec{\nabla}) \left(\frac{1}{r} \right)$$

$$\left. \begin{array}{l} \Rightarrow \nabla^2 \phi = \frac{1+\sigma}{4\pi Y (1-\sigma)} (\vec{f} \cdot \vec{\nabla}) \left(\frac{1}{r} \right) \end{array} \right\} \text{Div } \vec{v} = - \frac{1+\sigma}{2\pi Y} \vec{\nabla} \cdot \left(\frac{\vec{f}}{r} \right)$$

$$\Rightarrow \nabla^2 \phi = \frac{1+\sigma}{4\pi Y (1-\sigma)} (\vec{f} \cdot \vec{\nabla}) \left(\frac{1}{r} \right)$$

Trick 2 consider $\nabla^2 \psi = \frac{1}{r}$

$$\nabla^2 (\vec{f} \cdot \vec{\nabla}) \psi = \partial_{\beta\beta} f_{\mu} \partial_{\mu} \psi = f_{\mu} \partial_{\mu} \nabla^2 \psi = f_{\mu} \partial_{\mu} \left(\frac{1}{r} \right)$$

Hence
$$\nabla^2 \phi = \frac{1+\sigma}{4\pi Y (1-\sigma)} \nabla^2 (\vec{f} \cdot \vec{\nabla}) \psi$$

$$\Rightarrow \phi = \frac{1+\sigma}{4\pi Y (1-\sigma)} (\vec{f} \cdot \vec{\nabla}) \psi$$

Trick 3

$\nabla^2 \psi = \frac{1}{r}$, ψ is a function of r alone

$$\Rightarrow \frac{1}{r} \frac{d^2}{dr^2} (r\psi) = \frac{1}{r}$$

$$\Rightarrow \frac{d}{dr} (r\psi) = A + r$$

$$\Rightarrow r\psi = Ar + B + \frac{r^2}{2}$$

$$\Rightarrow \psi = A + \frac{B}{r} + \frac{r}{2}$$

↑
singular
at $r=0$

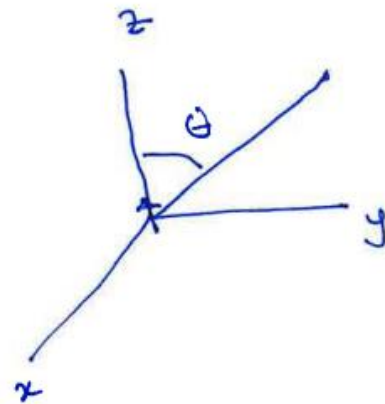
$\Rightarrow B=0$

$$\Rightarrow \boxed{\psi = A + r/2}$$

B Hence

$$\phi = \frac{1+\sigma}{4\pi\gamma(1-\sigma)} (\vec{f} \cdot \vec{\nabla}) \left(\frac{r}{2} \right)$$

$$\Rightarrow \phi = \frac{1+\sigma}{8\pi\gamma(1-\sigma)} f \cos\theta$$



Now work back ward:

$$\begin{aligned}
 \vec{u} &= \vec{u} + \vec{w} \\
 &= - \frac{2(1+\sigma)}{4\pi\gamma} \vec{f} \cdot \left(\frac{1}{r}\right) + \vec{\nabla} \phi \\
 &= - \frac{(1+\sigma)f}{2\pi\gamma r} \left[\hat{x} \frac{\cos\theta \sin\theta \cos\phi}{4(1-\sigma)} + \hat{y} \frac{\cos\theta \sin\theta \sin\phi}{4(1-\sigma)} \right. \\
 &\quad \left. + \hat{z} \frac{\sigma}{1-\sigma} \cos^2\theta \right]
 \end{aligned}$$

$$u \sim \frac{1}{r} \Rightarrow \text{strain} \sim \frac{1}{r^2}$$

$$G_{\alpha\beta}(\mathbf{r}) = \frac{1}{4\pi\mu} \left[\frac{\delta_{\alpha\beta}}{r} - \frac{1}{4(1-\sigma)} \frac{\partial^2 r}{\partial x_\alpha \partial x_\beta} \right]$$

How to solve Navier-Cauchy Eqn. ?

1. Identify the symmetry and write the Navier–Cauchy equation in a form that is most suited to your problem. Example: (7.26).
2. Reduce the PDE to one or more ODEs. Example (7.28). And solve this ODE. You will typically have two constants of integration.
3. From the solution calculate the strain. From strain calculate the stress. Using the (6.9) match the boundary conditions and calculate the constant. In some cases you may have a boundary condition directly on u itself, e.g., cases where elastic material is bounded by rigid boundary.
4. In some cases you may need to use different boundary conditions, e.g., the displacement must be zero and infinity (as I have done in deriving the Green's function), or the displacement must remain finite everywhere in the domain including the origin.