

(1)

What we know

The incompressible Navier-Stokes Eqn.

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} = \nu \nabla^2 \vec{v} - \nabla p / \rho; \quad \nabla \cdot \vec{v} = 0$$

Dynamical similarity: The solution depends only on

$$\boxed{Re = \frac{LU}{\nu}}$$

Reynolds Number

As $Re \rightarrow 0$,

$$\boxed{\begin{aligned} \nu \nabla^2 \vec{v} - \nabla p / \rho &= 0 \\ \nabla \cdot \vec{v} &= 0 \end{aligned}}$$

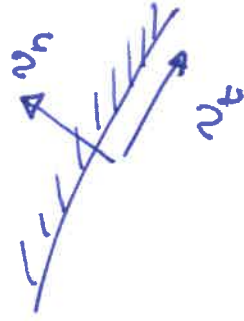
Stokes Equation

$$\text{As } Re \rightarrow \infty \quad \partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} = -\nabla p / \rho \quad \text{WRONG}$$

ν : kinematic viscosity

$$\mu = \rho \nu \equiv \text{dynamic viscosity}$$

Boundary conditions

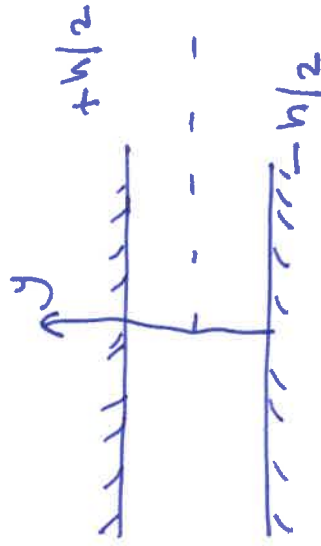


$$v_n = 0$$

$$v_t = 0$$

A simple case:

NO-SLIP boundary condition.



• $\leftarrow \Delta p \rightarrow$

~~By symmetry~~ : Choose

$$\Rightarrow \mu \frac{\partial^2 U}{\partial y^2} = -G$$

choose

$$U = U_0 \left(\frac{h^2}{4} - y^2 \right)$$

$$\Rightarrow U_0 = + \left(\frac{G}{2\mu} \right)$$

$$\mu \nabla^2 \vec{v} = \vec{\nabla} p, \quad v = (u, v)$$

$$\nabla^2 v_x = (\partial_x^2 + \partial_y^2) u \quad \vec{\nabla} p = (\partial_x p, 0)$$

$$\nabla^2 v_y = (\partial_x^2 + \partial_y^2) v$$

$$U(y), \quad v = 0, \quad \partial_x p = \text{constant} = -G$$

$$u = 0, \quad \text{at } y = \pm h/2$$

$$\Rightarrow \partial_y U = -2 U_0 y; \quad \partial_y^2 U = -2 U_0$$

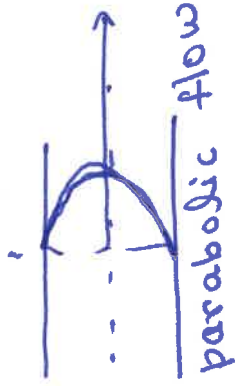
$$U = \left(\frac{+G}{2\mu} \right) \left(\frac{h^2}{4} - y^2 \right)$$

Flux of water : $Q = \int_{-h/2}^{+h/2} u(y) dy = u_0 \int_{-h/2}^{+h/2} \left(\frac{y^2}{4} - \frac{y^4}{8} \right) dy$

$$\left| \frac{1}{3} \cdot \frac{h^3}{8} \right|$$

$$= u_0 \left(\frac{y^2}{4} \cdot h - \frac{1}{3} \cdot \frac{h^3}{8} \right)$$

$$= u_0 \frac{h^3}{6} = \frac{u_0 h^3}{12}$$



x This is THE solution because of uniqueness.

Properties of Stokes Eqn.

1. Linear $\Rightarrow$

(a) Uniqueness

(b) Superposition (method of images etc)

2. No time $\Rightarrow$ The solution is set by the boundary condition at every Time.

No mixing.

(c) Symmetry of B.C. is reflected in solution.

4

Creeping flows $Re \ll 1$

Where does this apply?

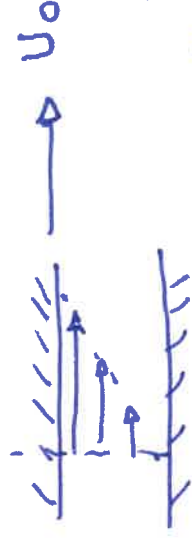
1. Microfluidics: $Re \sim 10^{-6}$

2. Swimming bacteria / sperm

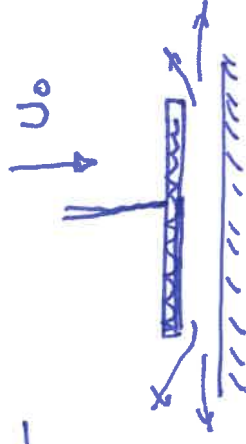
$Re \sim 10^{-4}$

3. Flows through porous rocks

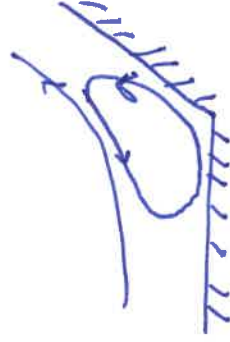
4. Plane-couette flow



5. Flow in narrow channels



6. Lubrication Problems



7. Flows around corners

8. Sedimenting sphere.

5

Lubrication approximation :

$$\mu \nabla^2 \vec{v} = \vec{\nabla} p$$

$$\vec{\nabla} \cdot \vec{v} = 0$$

$$\vec{v} = 0 \quad \text{at} \quad y = \pm h(x)$$

$h(x)$ is a slowly varying channel ; $\epsilon \equiv \frac{dh}{dx} \ll 1$

For a straight channel : $\epsilon = 0$, $U(y) = A(h^2 - y^2)$

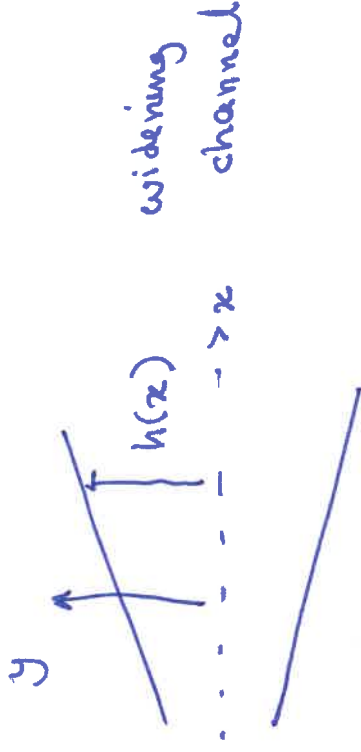
$$Q = \frac{4A}{3} h^3$$

A : depends on pressure gradient $= \frac{G}{\mu}$

Approximation : Flow remains locally parabolic

$$U(x, y) = A(x)(h^2 - y^2)$$

$$\Rightarrow Q = \frac{4}{3} A(x) h^3(x) = \text{constant} \quad (\text{because } \vec{\nabla} \cdot \vec{v} = 0)$$



(6)

$$\Rightarrow \frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial y^2} = -2\mu A(x)$$

$$\Rightarrow p(x) = p_{in} - 2\mu \int_0^x A(z) dz$$

$$= p_{in} - \frac{3}{2} \mu Q \int_0^x \frac{dz}{h^3(z)}$$

If you know the pressure difference

at length L , then

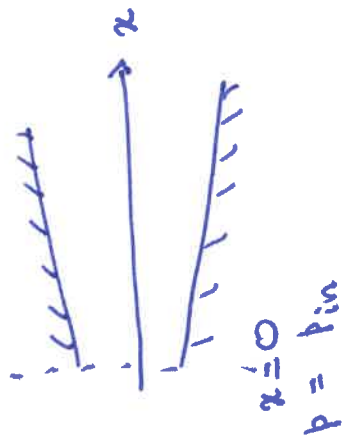
$$p_{out} - p_{in} = -\frac{3\mu Q}{2} \int_0^L \frac{dz}{h^3(z)}$$

$$\text{Then } A(x) = \frac{3Q}{4h^3}$$

; and

$$u(x, y) = A(x) \left(\frac{h^2}{2} - y^2 \right)$$

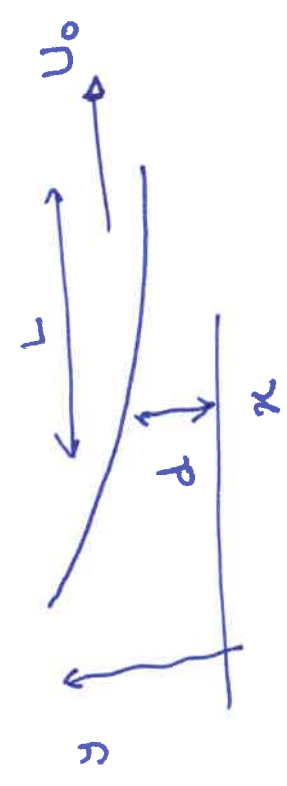
— Problem solved.



$p_{out} - p_{in}$ for a channel

7

Lubrication approximation (more formally)



$$\left. \begin{array}{l} \frac{\partial^2 u}{\partial y^2} \gg \frac{\partial^2 u}{\partial x^2} \\ \text{Locally parabolic} \end{array} \right\}$$

$$p = p(x)$$

$$V \sim \left(\frac{d}{L}\right) U \ll 1$$

$$U = 0 \text{ at } y=0, y=d$$

continuity Eqn.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Stokes Eqn.

$$\frac{\partial p}{\partial x} = \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} \right)$$

$$\Rightarrow u = \frac{G}{2\mu} y(d-y) + U_0 \left(\frac{d-y}{d} \right) \quad (1)$$

$$Q = \int_0^d u dy = \frac{G d^3}{12\mu} + \frac{1}{2} U_0 d = \text{constant}$$

8

$$\Rightarrow \frac{dp}{dx} = -G = \mu \frac{\partial^2 U}{\partial y^2} = 6\mu \left(\frac{U_0}{d^2} - \frac{2Q}{d^3} \right) \quad (2)$$

$$\Rightarrow p(x) - p_{in} = 6\mu \int_0^x \left(\frac{U_0}{d^2} - \frac{2Q}{d^3} \right) dx$$

Eliminate G from (1) & (2) to obtain

$$U = U(d) = \frac{(d-y)(d-3y)}{d^2} + Q \frac{6y(d-y)}{d^3}$$

obtain V from the continuity eqn.

$$\text{To calculate drag} = \int_0^L \sigma_{xy}(x) dx \quad (L_z) = \mu \int_0^L \frac{\partial U}{\partial y} \bigg|_{y=0} dx \cdot L_z$$

$$\text{Lift} = \int_0^L G(x) dx \quad (L_z)$$