

Summary of what we know

1. Tensors : $T_{\alpha\beta} = \alpha_{\alpha\mu} \alpha_{\beta\nu} T_{\mu\nu}$ 2. Any small deformation of a soft body can be ~~describ~~ described by

Isotropic tensors:

$$E_{\alpha\beta\mu\nu} = \lambda (\delta_{\alpha\beta} \delta_{\mu\nu}) + \mu (\delta_{\alpha\mu} \delta_{\beta\nu} + \delta_{\alpha\nu} \delta_{\beta\mu}) + \nu (\delta_{\alpha\beta} \delta_{\mu\nu} - \delta_{\alpha\nu} \delta_{\beta\mu})$$

$$(\Delta r)'_{\alpha} = (\Delta r)_{\alpha} + \epsilon_{\alpha\beta\gamma} \Omega_{\beta} (\Delta r)_{\gamma} + S_{\alpha\beta} (\Delta r)_{\beta}$$

with $\vec{u}$ being the deformation field

$$\text{and } \Omega_{\alpha} = \epsilon_{\alpha\beta\gamma} \partial_{\beta} u_{\gamma} \quad \text{anti-symmetric}$$

$$S_{\alpha\beta} = \frac{1}{2} (\partial_{\alpha} u_{\beta} + \partial_{\beta} u_{\alpha}) \quad \text{symmetric}$$

3. The equation of equilibrium

$$\partial_{\beta} \sigma_{\alpha\beta} + \underline{f_{\alpha}} = 0 \quad \text{force per unit vol}$$

with boundary condition

$$\sigma_{\alpha\beta} n_{\beta} = \underline{f_{\alpha}^{\text{surf}}} \quad \leftarrow \begin{array}{l} \text{force per unit} \\ \text{area} \end{array}$$

$$F = \frac{1}{2} \lambda S_{kk}^2 + \mu S_{\alpha\beta} S_{\alpha\beta}$$

$$= \frac{1}{2} K \frac{1}{3} \delta_{\alpha\beta} S_{kk}^2 + \mu \left(S_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} S_{kk} \right)^2$$

$$\sigma_{\alpha\beta} = \left(\frac{\partial F}{\partial S_{\alpha\beta}} \right)_T = K \delta_{\alpha\beta} S_{kk} + 2\mu \left(S_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} S_{kk} \right)$$

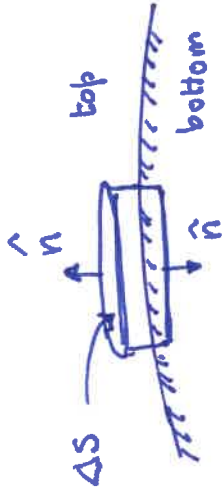
$$(\lambda + \mu) \vec{\nabla} (\vec{\nabla} \cdot \vec{u}) + \mu \vec{\nabla}^2 \vec{u} = 0$$

$$K > 0, \mu > 0$$

Because of thermodynamic stability.

Navier - Cauchy Eqn.

Boundary condition on interfaces



$$\int_V \partial_\beta \sigma_{\alpha\beta} dV = \int f dV$$

The sides do not contribute.

$$(\sigma_{\alpha\beta}^t - \sigma_{\alpha\beta}^b) n_\beta \Delta S = \int_{\text{surf}} f \Delta S$$

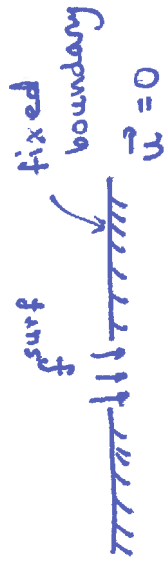
If there are no surface density of forces,

$$(\sigma_{\alpha\beta}^t - \sigma_{\alpha\beta}^b) n_\beta = 0$$

The normal component of stress is continuous. If there is vacuum outside, $\sigma_{\alpha\beta}^t = 0$

$$\Rightarrow \sigma_{\alpha\beta}^b n_\beta = 0 \Rightarrow$$

A condition on $\Rightarrow$ condition on $\frac{\partial u_\alpha}{\partial x_\beta} n_\beta$ → Neumann condition



Dirichlet boundary condition.

Properties of solution

Linear Equation

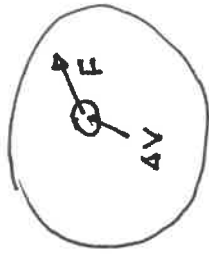
$$\vec{\nabla}(\vec{\nabla} \cdot \vec{u}) + (1 - 2\sigma) \vec{\nabla}^2 \vec{u} = 0$$

- Uniqueness
- Symmetry
- Superposition

Green's function for Navier-Cauchy equation:

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Egn. of equilibrium:



EXAMINATION

$$F_\alpha + \partial_\beta \sigma_{\alpha\beta} \Delta V = 0$$

Define force per unit volume $\vec{f} = \frac{\vec{F}}{\Delta V}$

$$\Rightarrow f_\alpha + \partial_\beta \sigma_{\alpha\beta} = 0$$

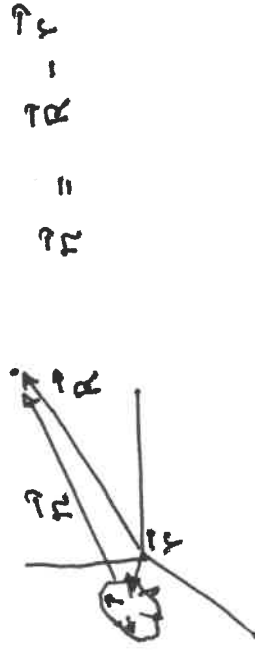
From this: $\frac{1}{1-2\sigma} \vec{\nabla}(\text{div } \vec{u}) + \vec{\nabla} \vec{u} = - \frac{2(1+\sigma)}{Y} \vec{f}$

$\mathcal{L} \vec{u} = \vec{f}$ $\mathcal{L}$: linear operator

or
$$u_\alpha(\vec{R}) = \int G_{\alpha\beta}(\vec{R} - \vec{r}) f_\beta(\vec{r}) d\vec{r}$$

Green's function

In this case a 2nd rank tensor



what eqn. does G satisfy? $\mathcal{L} u = f$;

$$u(x) = \int G(x-y) f(y) dy$$

$$\Rightarrow \mathcal{L} u = \int \mathcal{L} G(x-y) f(y) dy = f(x)$$

$$\Rightarrow \boxed{\mathcal{L}(x) G(x-y) = \delta(x-y)}$$

Solve:

$$(1-2\sigma)^{-1} \vec{\nabla} (\text{div } \vec{u}) + \nabla^2 \vec{u} = -\frac{2(1+\sigma)}{Y} \vec{f} \delta(\vec{r})$$

Remember: electrostatic potential for a point charge

$$\text{solution: } U(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r}$$

$$\vec{u} = \vec{u} + \vec{w} \quad \text{such that}$$

Let us choose:

$$(1-2\sigma)^{-1} \vec{\nabla} \text{div} (\vec{u} + \vec{w}) + \nabla^2 (\vec{u} + \vec{w}) = 0$$

Take a curl:

$$\nabla^2 (\vec{\nabla} \times \vec{w}) = 0$$

~~test~~

$$\vec{\nabla} \times \vec{w} = 0$$

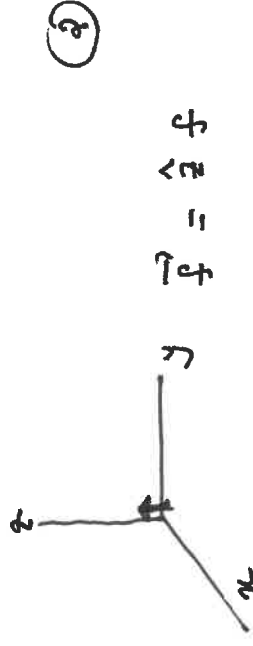
$\Rightarrow$

$$\vec{w} = \vec{\nabla} \phi$$

$\Rightarrow$

$$(1-2\sigma)^{-1} \vec{\nabla} (\text{div } \vec{\nabla} \phi) + \nabla^2 (\vec{\nabla} \phi) = -(1-2\sigma)^{-1} \vec{\nabla} \text{div } \vec{u}$$

$$\frac{2(1-\sigma)}{(1-2\sigma)} \vec{\nabla} (\vec{\nabla} \phi) = -(1-2\sigma)^{-1} \vec{\nabla} \text{div } \vec{u}$$



$$\nabla^2 U = \frac{Q \delta(\vec{r})}{\epsilon_0} \quad \text{Poisson eqn.}$$

$$\nabla^2 \vec{u} = -\frac{2(1+\sigma)}{Y} \vec{f} \delta(\vec{r})$$

$\rightarrow$

Each component like Poisson eqn.

$$\vec{u} = -\frac{2(1+\sigma)}{Y 4\pi} \vec{f} \left(\frac{1}{r}\right) = \left[-\frac{2(1+\sigma)}{2\pi Y} \frac{\vec{f}}{r} \right]$$

$\vec{u}$ points in the same direction as $\vec{f}$

but falls off a $1/r$ from the origin.

$$\Rightarrow 2(1-\sigma) \nabla^2 (\vec{\nabla} \phi) = - \vec{\nabla} \operatorname{div} \vec{v}$$

$$\operatorname{div} \vec{v} = - \frac{1+\sigma}{2\pi Y} \partial_x \left(\frac{f(x)}{r} \right)$$

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$$= \frac{(1+\sigma)}{2\pi Y} \vec{\nabla} (\vec{f} \cdot \vec{\nabla}) \left(\frac{1}{r} \right)$$

$$\vec{\nabla} (\nabla^2 \phi) = \frac{(1+\sigma)}{4\pi Y (1-\sigma)} \vec{\nabla} [\vec{f} \cdot \vec{\nabla} \left(\frac{1}{r} \right)]$$

$\Rightarrow$

$$\nabla^2 \phi = \frac{(1+\sigma)}{4\pi Y (1-\sigma)} (\vec{f} \cdot \vec{\nabla}) \left(\frac{1}{r} \right)$$

$\Rightarrow$

$$\nabla^2 \phi = \frac{(1+\sigma)}{4\pi Y (1-\sigma)} \nabla^2 (\vec{f} \cdot \vec{\nabla}) \left(\frac{1}{r} \right) \phi$$

$\Rightarrow$

$$\phi = \frac{1+\sigma}{4\pi Y (1-\sigma)} (\vec{f} \cdot \vec{\nabla}) \phi$$

$\Rightarrow$

$$\phi = \frac{1+\sigma}{4\pi Y (1-\sigma)} (\vec{f} \cdot \vec{\nabla}) \left(\frac{r}{2} \right)$$

$\Rightarrow$

$$\text{consider } \nabla^2 \psi = \frac{1}{r}$$

$$\nabla^2 (\vec{f} \cdot \vec{\nabla}) \psi$$

$$= \partial_{\mu} f_{\mu} \partial_{\mu} \psi$$

$$= f_{\mu} \partial_{\mu} (\nabla^2 \psi) = (\vec{f} \cdot \vec{\nabla}) \left(\frac{1}{r} \right)$$

$$\text{solve } \nabla^2 \psi = \frac{1}{r}$$

ψ is a function of r alone

$$\Rightarrow \frac{1}{r} \frac{d^2}{dr^2} (r\psi) = \frac{1}{r}$$

$$\Rightarrow \frac{d}{dr} (r\psi) = A + r$$

$$r\psi = Ar + B + \frac{r^2}{2}$$

$\Rightarrow$

$$\psi = A + \frac{B}{r} + \frac{r}{2}$$

$\Rightarrow$

$\uparrow$

singular at $r=0$

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$$\Rightarrow \phi = \frac{1+\sigma}{8\pi\gamma(1-\sigma)} f \cos \theta$$

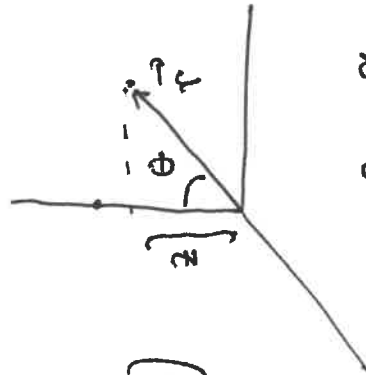
$$\Rightarrow \vec{\omega} = \vec{\nabla} \phi$$

$$= \frac{(1+\sigma)}{8\pi\gamma(1-\sigma)} f (\hat{x} \partial_x + \hat{y} \partial_y + \hat{z} \partial_z) \left(\frac{r}{z} \right)$$

$$= \left(\begin{array}{c} \hat{x} \\ -\hat{y} \\ \hat{z} \end{array} \right) \left[\frac{\partial}{\partial x} \left(\frac{r \cos \theta}{r^2} \right) + \hat{y} \frac{\partial}{\partial y} \left(\frac{r \cos \theta}{r^2} \right) + \hat{z} \frac{\partial}{\partial z} \left(\frac{r \cos \theta}{r^2} \right) \right]$$

$$= \left(\begin{array}{c} \hat{x} \\ -\hat{y} \\ \hat{z} \end{array} \right) \left[\frac{\partial}{\partial x} \left(\frac{r \cos \theta}{r^2} \right) + \hat{y} \frac{\partial}{\partial y} \left(\frac{r \cos \theta}{r^2} \right) + \hat{z} \frac{\partial}{\partial z} \left(\frac{r \cos \theta}{r^2} \right) \right]$$

$$\begin{aligned} \frac{\partial}{\partial x} \left(\frac{r}{z} \right) &= \frac{1}{z} - \frac{r}{z^2} \frac{\partial z}{\partial x} = \frac{1}{z} - \frac{r \cos \theta}{z^2} \\ \frac{\partial}{\partial y} \left(\frac{r}{z} \right) &= \frac{1}{z} - \frac{r}{z^2} \frac{\partial z}{\partial y} = \frac{1}{z} - \frac{r \sin \theta}{z^2} \\ \frac{\partial}{\partial z} \left(\frac{r}{z} \right) &= \frac{1}{z} - \frac{r}{z^2} \frac{\partial z}{\partial z} = \frac{1}{z} - \frac{r}{z^2} \end{aligned}$$



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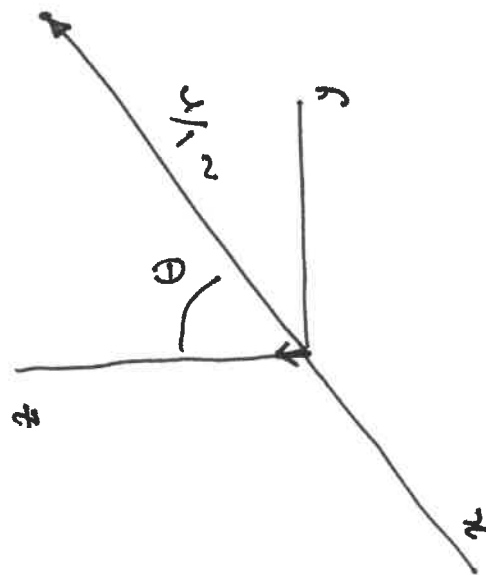
$$\begin{aligned} \Rightarrow \vec{\omega} &= \left(\frac{\partial}{\partial x} \cdot \vec{\nabla} \right) f = f \frac{\partial}{\partial z} \vec{z} \\ \frac{\partial}{\partial x} \left(\frac{r}{z} \right) &= \frac{1}{z} - \frac{r}{z^2} \frac{\partial z}{\partial x} = \frac{1}{z} - \frac{r \cos \theta}{z^2} \\ \frac{\partial}{\partial y} \left(\frac{r}{z} \right) &= \frac{1}{z} - \frac{r}{z^2} \frac{\partial z}{\partial y} = \frac{1}{z} - \frac{r \sin \theta}{z^2} \\ \frac{\partial}{\partial z} \left(\frac{r}{z} \right) &= \frac{1}{z} - \frac{r}{z^2} \frac{\partial z}{\partial z} = \frac{1}{z} - \frac{r}{z^2} \end{aligned}$$

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$$\begin{aligned} \sigma_{xx} = & -\frac{(1+\sigma)}{2\pi\gamma} f \frac{z}{r} + \frac{(1+\sigma)}{8\pi\gamma(1-\sigma)} r \left[-\hat{x} \cos\theta \sin\theta \cos\phi \right. \\ & \left. - \hat{y} \cos\theta \sin\theta \sin\phi \right. \\ & \left. + \hat{z} \sin^2\theta \right] \end{aligned}$$

$$\begin{aligned} = & \frac{(1+\sigma)}{2\pi\gamma} r \left[-\frac{\hat{x} \cos\theta \sin\theta \cos\phi}{4(1-\sigma)} - \frac{\hat{y} \cos\theta \sin\theta \sin\phi}{4(1-\sigma)} \right. \\ & \left. + \left(\frac{\sigma}{1-\sigma}\right)(\sin^2\theta - 1) \right] \end{aligned}$$

$$\begin{aligned} = & -\frac{(1+\sigma)}{2\pi\gamma} r \left[+\frac{\hat{x} \cos\theta \sin\theta \cos\phi}{4(1-\sigma)} + \frac{\hat{y} \cos\theta \sin\theta \sin\phi}{4(1-\sigma)} \right. \\ & \left. + \left(\frac{\sigma}{1-\sigma}\right) \cos^2\theta \right] \end{aligned}$$



falls off a electrostatic

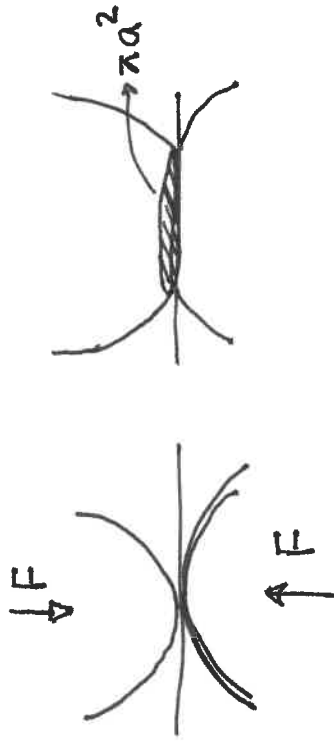
potential.

$\Rightarrow$ strain $\sim \frac{1}{r^2}$ Like electrostatic field.

Hertzian contact mechanics

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simplest case : Two spheres of radius R_1 and R_2 are pushed towards each other with force F . They are made with material of Young's modulus Y , and Poisson ratio ν . ~~What~~ How much does the centres move ~~to~~ towards each other?

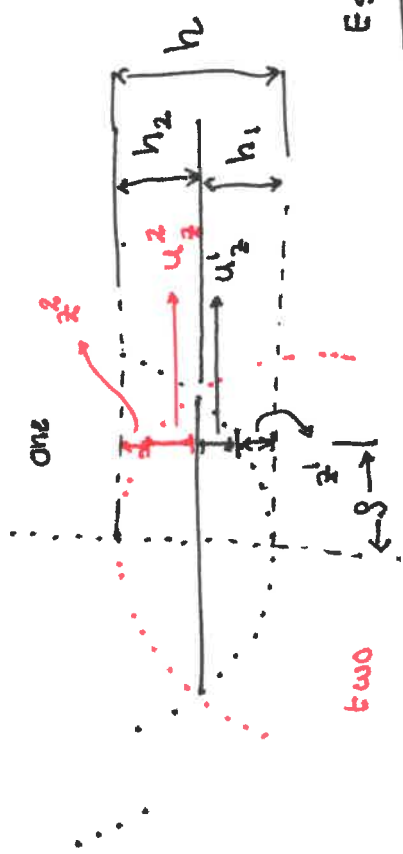


The spheres will have a contact area.

Let us by symmetry assume it to be a circle of radius a .

Let us assume that the deformation remains small everywhere and remains limited to a volume of $\pi a^2 h_1$ in sphere 1 and $\pi a^2 h_2$ in sphere 2. The distance moved is $h = h_1 + h_2$

As the spheres are made of different material h_1 and h_2 are not the same.



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$$h_1 = u_z^1 + z^1 = u_z^1 + s^2 \left(\frac{1}{2R_1} \right)$$

$$h_2 = u_z^2 + z^2 = u_z^2 + s^2 \left(\frac{1}{2R_2} \right)$$

Estimate $s \sim a$

$$\text{Strain} \sim \frac{u_z^1(s) - h_1}{s} \sim \frac{h_1}{a}$$

$$\frac{u_z^2(s) - h_2}{s} \sim \frac{h_2}{a}$$

vertical stress, $\sigma \sim Y(\text{strain})$

$\Rightarrow \sigma_1 \sim Y_1 \frac{h_1}{a}, \quad \sigma_2 \sim Y_2 \frac{h_2}{a}$

But the stress must balance.

$$\sigma_1 \sim \sigma_2 \sim \sigma = \frac{F}{\pi a^2}$$

$$\Rightarrow \left(\frac{h_1}{a} \right) \sim \frac{\sigma}{Y_1} \sim \frac{F}{\pi a^2 Y_1}$$

$$\left(\frac{h_2}{a} \right) \sim \frac{\sigma}{Y_2} \sim \frac{F}{\pi a^2 Y_2}$$

From geometry

$$h = u_z^1 + \frac{s^2}{2R_1} + u_z^2 + \frac{s^2}{2R_2}$$

$$\Rightarrow (h_1 - u_z^1) + (h_2 - u_z^2) = s^2 \left(\frac{1}{2R_1} + \frac{1}{2R_2} \right)$$

$$\Rightarrow \frac{h_1}{a} + \frac{h_2}{a} \sim a \left(\frac{1}{2R_1} + \frac{1}{2R_2} \right)$$

$$\Rightarrow \frac{F}{\pi a^2} \left(\frac{1}{Y_1} + \frac{1}{Y_2} \right) = \frac{a}{2} \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

$$\Rightarrow F = (\pi a^3) \frac{Y_h}{R_h}$$

$$\Rightarrow a^3 = \frac{R_h Y_h}{\pi} F$$

$$a = \left(\frac{R_h}{\pi Y_h} \right)^{1/3} F^{1/3}$$

The contact area goes as one-third power of F

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$$h = h_1 + h_2 = \frac{F}{\pi a \gamma_1} + \frac{F}{\pi a \gamma_2} = \frac{F}{\pi a} \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2} \right) = \frac{F}{\pi a \gamma_h}$$

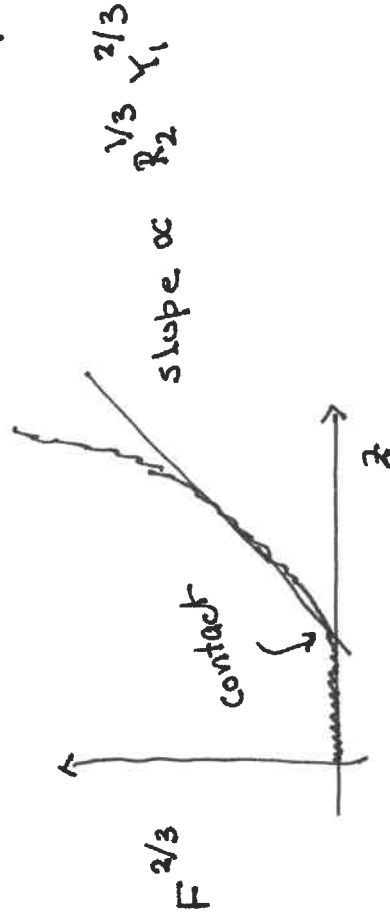
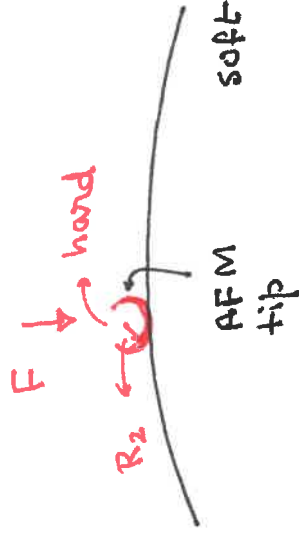
$$= \frac{F}{\pi \gamma_h} \frac{(\pi \gamma_h)^{1/3}}{R_h^{1/3}} \frac{1}{F^{1/3}} = \left(\frac{F}{\pi} \right)^{2/3} \frac{1}{\gamma_h^{2/3} R_h^{1/3}}$$

Now consider, for two: very hard material & $\gamma_2 \rightarrow \infty$

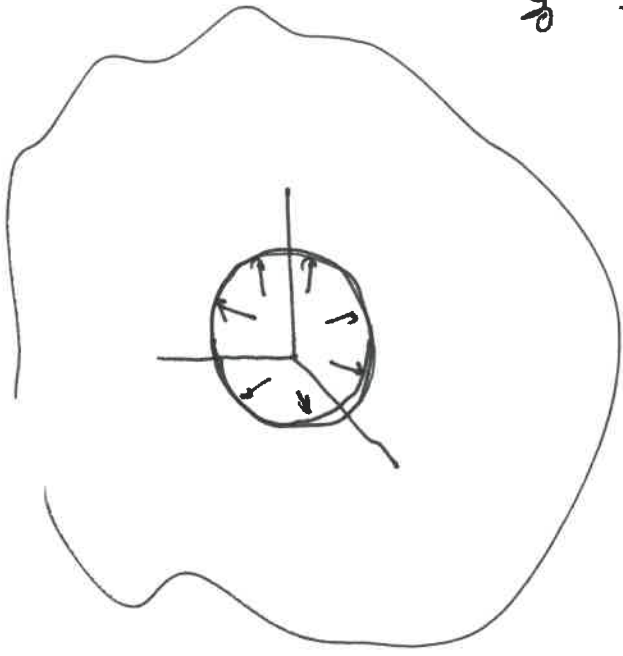
for one: very large radius of curvature, $R_1 \rightarrow \infty$

$$\Rightarrow \gamma_h = \gamma_1, \quad R_h = R_2$$

$$\Rightarrow h = \left(\frac{F}{\pi} \right)^{2/3} \frac{1}{\gamma_1^{2/3} R_2^{1/3}}$$



In practice, the result is obtained from exact geometry of the tip and numerical solution of Navier-Cauchy equation.



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Inside an elastic medium there is a sphere of radius a . Through this spherical hole uniform hydrostatic pressure acts outward. Calculate the deformation field everywhere. Assume a to be small.

Two solid balls collide with relative velocity v . Calculate the time they stay in contact.