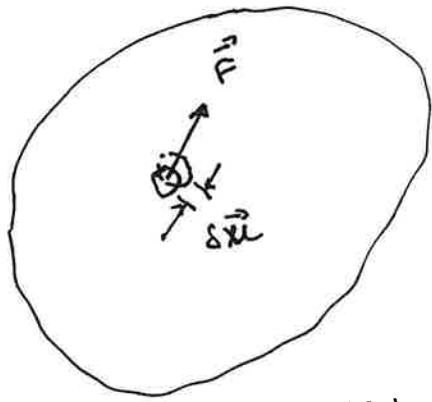


Elastic free energy



Deformation field $\vec{u}(\vec{x})$

vector field

strain field $S_{\beta\mu}(\vec{x})$

tensor field.

$$\delta W = \vec{F} \cdot \delta \vec{u}$$

$$= F_{\beta} \delta u_{\beta} = \partial_{\mu} \sigma_{\beta\mu} \Delta V \delta u_{\beta} \quad \text{virtual displacements}$$

$$\text{Total work done : } \int (\partial_{\mu} \sigma_{\beta\mu}) \delta u_{\beta} dV$$

$$= \int \frac{\partial}{\partial x_{\mu}} (\sigma_{\beta\mu} \delta u_{\beta}) dV - \int \sigma_{\beta\mu} \frac{\partial}{\partial x_{\mu}} (\delta u_{\beta}) dV$$

$$= \oint \sigma_{\beta\mu} \delta u_{\beta} n_{\mu} dS - \int \sigma_{\beta\mu} \delta \left(\frac{\partial u_{\beta}}{\partial x_{\mu}} \right) dV$$

$$= 0$$

$$- \frac{1}{2} \int \sigma_{\beta\mu} \delta \left(\frac{\partial u_{\beta}}{\partial x_{\mu}} + \frac{\partial u_{\mu}}{\partial x_{\beta}} \right) dV$$

virtual displacements

are zero on the surface

$$= - \int \sigma_{\beta\mu} \delta S_{\beta\mu} dV \Rightarrow$$

$$\boxed{\delta W = \sigma_{\beta\mu} \delta S_{\beta\mu}}$$

Thermodynamics

$$T dS = dU + dW$$

heat ↗
↑
work ↖

internal

energy

$$\Rightarrow T dS = dU - \sigma_{\alpha\beta} ds_{\alpha\beta}$$

Helmholtz free energy

$$F = U - TS$$

$$\Rightarrow dF = dU - T dS - S dT$$

$$= \sigma_{\alpha\beta} ds_{\alpha\beta} - S dT$$

$$\Rightarrow \boxed{\sigma_{\alpha\beta} = \left(\frac{\partial F}{\partial s_{\alpha\beta}} \right)_T}$$

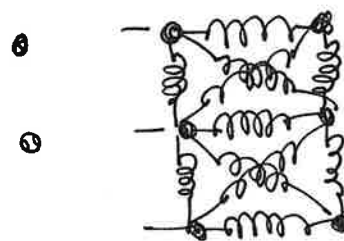
For liquids $\sigma_{\alpha\beta} = -p \delta_{\alpha\beta}$ ↗ pressure

$$\begin{aligned} \Rightarrow T dS &= dU + p \delta_{\alpha\beta} ds_{\alpha\beta} \\ &= dU + p ds_{\mu/\nu} \\ &= dU + p dV \end{aligned}$$

which you remember from your thermodynamics.

How to write down the free energy?

Start with a model:



Hamiltonian $\mathcal{H}(p_i, q_i)$

Calculate partition function

$$Z = \int \exp(-\beta \mathcal{H}) \prod_{i=1}^N dp_i dq_i$$

$$F = -\frac{1}{\beta} \ln Z$$

- Practically almost impossible to do.
- We obtain a model-dependent result not an insight.

Let us obtain F by "pure thinking"

ok reduce our ambition, find how F depends on $s_{\alpha\beta}$

1. For no strain $F = F_0$ some constant.

why?

because neither translation nor rotation changes the relative position of the molecules thus the F cannot depend on them must depend only on $s_{\alpha\beta}$

2. Expand F in a Taylor series

$$F = F_0 + A s_{\alpha\beta} + B s_{\alpha\beta} s_{\mu\nu} + \text{h.o.t}$$

$s_{\alpha\beta} = 0$ must be a minima of the free energy.
otherwise our soft body is not stable.

$$\Rightarrow A = 0$$

3.

$$F = F_0 + B s_{\alpha\beta} s_{\mu\nu} + \text{h.o.t.}$$

must be a 4th rank tensor $E_{\alpha\beta\mu\nu}$

otherwise F is not a scalar.

Hence

$$F = F_0 + E_{\alpha\beta\mu\nu} s_{\alpha\beta} s_{\mu\nu} + \text{h.o.t.}$$

For an isotropic material

$$E_{\alpha\beta\mu\nu} = \lambda s_{\alpha\beta} s_{\mu\nu} + \mu (\delta_{\alpha\mu} \delta_{\beta\nu} + \delta_{\alpha\nu} \delta_{\beta\mu}) + \nu (\delta_{\alpha\mu} \delta_{\beta\nu} - \delta_{\alpha\nu} \delta_{\beta\mu})$$

(6)

Isotropic tensors :

Tensors that look the same in all rotated coordinate system.

$$T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

construct the quadratic form

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This must be a surface that looks the same in all rotated coordinates.

$\Rightarrow$ It must be a sphere.

$$T = A \delta_{\alpha\beta}$$

The only 2nd rank isotropic tensor.

(2)

$$dF = K s_{\mu\mu} ds_{\mu\mu} + 2\mu \left(s_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} s_{\mu\mu} \right) d \left(s_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} s_{\mu\mu} \right)$$

$$= \left[K \delta_{\alpha\beta} s_{\mu\mu} + 2\mu \left(s_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} s_{\mu\mu} \right) \right] ds_{\alpha\beta}$$

$$\sigma_{\alpha\beta} = \left(\frac{\partial F}{\partial s_{\alpha\beta}} \right)_T = K \delta_{\alpha\beta} s_{\mu\mu} + 2\mu \left(s_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} s_{\mu\mu} \right)$$

stress is linearly related to strain.

Hook's law.

Summary: Elastic free energy at constant temperature:

$$K = \lambda + \frac{2\mu}{3}$$

$$F = \frac{1}{2} \lambda s_{\alpha\alpha}^2 + \mu s_{\alpha\beta} s_{\alpha\beta} = \frac{1}{2} G s_{\alpha\beta} s_{\alpha\beta}$$

$$\sigma_{\alpha\beta} = K \delta_{\alpha\beta} s_{\mu\mu} + 2\mu s_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} s_{\mu\mu}$$