

~~Flu~~ what we know:

$$\partial_\beta \nabla_\alpha p = 0 \quad \text{static equilibrium}$$

①

$$\rho \frac{dV_\alpha}{dt} = \partial_\beta \nabla_\alpha p$$

Equation of motion of a soft body.

- Fluid particle, density and velocity in coarse grained sense.
- A fluid cannot sustain a shear strain. For a fluid

$$(\text{stress}) = -p \delta_{\alpha\beta} + \left( \begin{array}{l} \text{Other terms} \\ \text{we do not worry about} \\ \text{now} \end{array} \right)$$

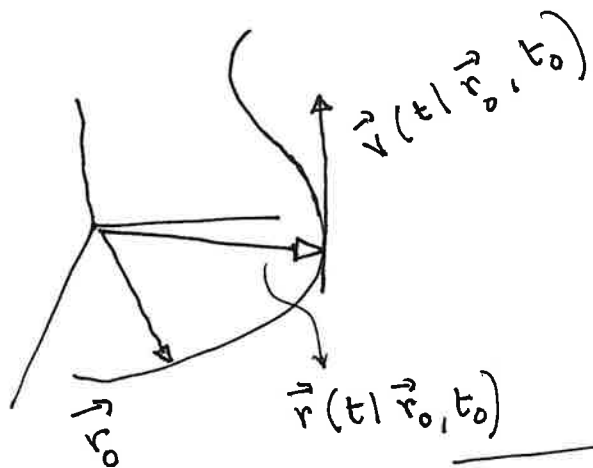
$$\Rightarrow \rho \frac{dV_\alpha}{dt} = -\partial_\alpha p + \rho g_\alpha$$

or

$$\left[ \rho \frac{d\vec{V}}{dt} = -\vec{\nabla} p + \rho \vec{g} \right]$$

Not a field theory.

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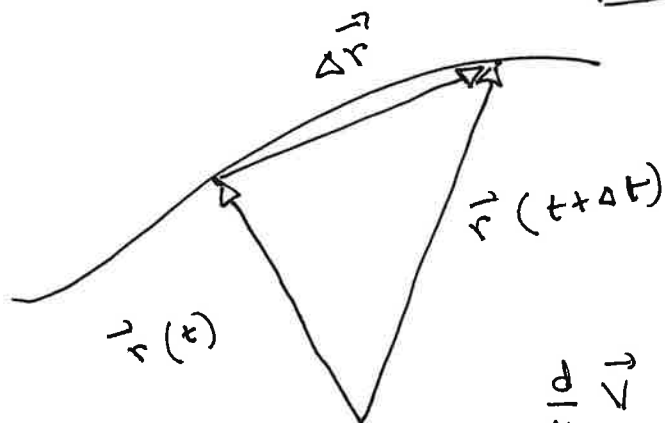


$$\vec{v}(t | \vec{r}_0, t) = \frac{d}{dt} \vec{r}(t | \vec{r}_0, 0)$$

$$\text{or } \vec{r} = \vec{r}_0 + \int_0^t \vec{v}(s | \vec{r}_0, 0) ds$$

$$\boxed{\vec{v}(t | \vec{r}_0, 0) = \vec{v}(\vec{r}, t)}$$

Lagrangian displacement



$$\vec{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t}$$

$$\frac{\vec{v}(t + \Delta t | \vec{r}_0, 0) - \vec{v}(t | \vec{r}_0, 0)}{\Delta t}$$

$$\frac{d}{dt} \vec{v} = \lim_{\Delta t \rightarrow 0}$$

$$\vec{v} \approx \vec{v}(t) \Delta t \quad \Delta \vec{r} = \vec{v}(t) \Delta t$$

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$$\vec{V}(t + \Delta t, \vec{r}_0) = \vec{V}(\vec{r} + \Delta \vec{r}, t + \Delta t)$$

$$\vec{V}(t, \vec{r}_0) = \vec{V}(\vec{r}, t)$$

Now  $\vec{V}(\vec{r} + \Delta \vec{r}, t + \Delta t) = \vec{V}(\vec{r}, t)$

$$+ \left( \frac{\partial v_x}{\partial x} \Delta x + \frac{\partial v_x}{\partial y} \Delta y + \frac{\partial v_x}{\partial z} \Delta z + \frac{\partial v_x}{\partial t} \Delta t \right) \hat{x}$$

$$+ \left( \frac{\partial v_y}{\partial x} + \dots \right)$$

$$+ \left( \frac{\partial v_z}{\partial x} + \dots \right) \hat{z}$$

$\Rightarrow \vec{V} \neq$

$$\Rightarrow v_\alpha(\vec{r} + \Delta\vec{r}, t + \Delta t) - v_\alpha(\vec{r}, t)$$

$$= \sum_{\beta=1}^3 \frac{\partial v_\alpha}{\partial x_\beta} \Delta x_\beta + \frac{\partial v_\alpha}{\partial t} \Delta t + \text{h.o.t.}$$

$$\Delta x_\beta = v_\beta \Delta t$$

$$= \sum_{\beta=1}^3 \frac{\partial v_\alpha}{\partial x_\beta} v_\beta \Delta t + \frac{\partial v_\alpha}{\partial t} \Delta t + \text{h.o.t.}$$

$$= \left( \frac{\partial v_\alpha}{\partial t} + v_\beta \frac{\partial v_\alpha}{\partial x_\beta} \right) \Delta t + \text{h.o.t.}$$

$$\Rightarrow \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v}$$

$$\Rightarrow \rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right] = -\vec{\nabla} p + \rho \vec{g}$$

$$\Rightarrow \boxed{\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} p + \vec{g}} \quad \text{EULER egn.}$$

# Comments

- Advective derivative
- Geometric in origin
- Nonlinear

• Ideal, inviscid, & dry water.

Bernoulli's theorem:  $\vec{g} = -\vec{\nabla}\Phi$  : gravitational potential.

Assume steady flow.

$$\partial_t \vec{v} = 0$$

$$\Rightarrow \left( \vec{v} \cdot \vec{\nabla} \right) \vec{v} = - \vec{\nabla} \left( \frac{g}{\rho} + \Phi \right)$$

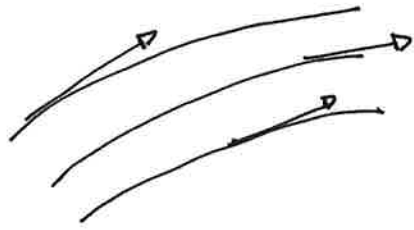
$$\Rightarrow \left( \vec{\nabla} \times \vec{v} \right) \times \vec{v} + \frac{1}{2} \vec{\nabla} \left( \vec{v} \cdot \vec{v} \right) = - \vec{\nabla} \left( \frac{g}{\rho} + \Phi \right)$$

Take a dot product with  $\vec{v}$

$$\Rightarrow \vec{v} \cdot \left( \vec{\nabla} \times \vec{v} \right) \times \vec{v} + \frac{1}{2} \vec{v} \cdot \vec{\nabla} \left( \vec{v} \cdot \vec{v} \right) = - \vec{v} \cdot \vec{\nabla} \left( \frac{g}{\rho} + \Phi \right) \cdot \vec{v}$$

$$\Rightarrow 0 = \vec{v} \cdot \vec{\nabla} \left( \frac{1}{2} \vec{v}^2 + \frac{g}{\rho} + \Phi \right)$$

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streamlines: Tangent at any point gives the velocity at that point.

For a steady flow  $\frac{v^2}{2} + \frac{p}{\rho} + \bar{\Psi}$  is constant on streamlines.

Note:

(1) Different constants on different streamlines.

(2) Only ~~use~~ when density is constant

(3)  $\rho \frac{v^2}{2} + p + \rho \bar{\Psi} = \text{constant}$

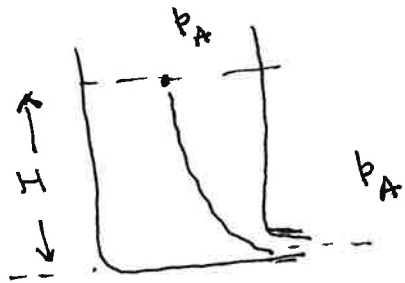
— statement of conservation of energy.

(4) If  $\vec{\nabla} \times \vec{v} = 0$ ; then  $\frac{v^2}{2} + \frac{p}{\rho} + \bar{\Psi} = 0$  everywhere the same constant

Potential flow.

What can Bernoulli do?

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$$\frac{p}{A} + H \rho g = \rho \frac{v^2}{2} + \frac{p}{A}$$

$\Rightarrow$

$$v = \sqrt{2gH}$$

Toricelli's law.

But flow rate  $Q \neq A v$   
because  $v$  is not constant over the area of the hole.

• Can Bernoulli fly airplanes?