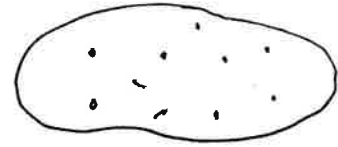


Conservation Law

Global: Total number of particles in a volume V is constant



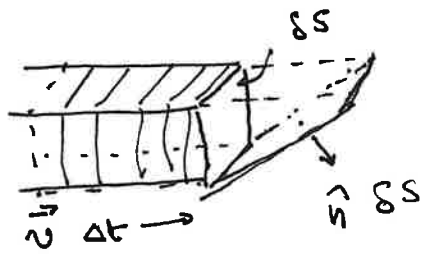
$$\int_V \rho \, dV = \text{constant}$$

$$\int_V \partial_t \rho \, dV = 0$$

Local: Total number of particles in any sub-volume of V , δV , plus the number that enters, minus the number that goes out is constant

$$\int_{\delta V} \rho \, dV + \left(\text{mass that enters } \delta V \right) - \left(\text{mass that leaves } \delta V \right) = \text{constant}$$

$$\Rightarrow \int_{\delta V} \partial_t \rho \, dV + \left(\text{rate of mass entering } \delta V \right) - \left(\text{rate of mass leaving } \delta V \right) = 0$$



(2)

Definition of mass flux.
 $\vec{J} \cdot \hat{n} \delta S$

Rate at which mass leaves through $\hat{n} \delta S$

$$\Rightarrow \int_{\delta V} \frac{\partial \rho}{\partial t} dV + \oint_{\delta S} \vec{J} \cdot \hat{n} dS = 0$$

$$\Rightarrow \int_{\delta V} \left(\frac{\partial \rho}{\partial t} + \text{div} \vec{J} \right) dV = 0$$

$$\Rightarrow \left(\frac{\partial \rho}{\partial t} + \text{div} \vec{J} \right) = 0$$

$$\vec{J} = \rho \vec{v}$$

continuity eqn.

$$\Rightarrow \frac{\partial \rho}{\partial t} + \text{div} (\rho \vec{v}) = 0$$

$$\Rightarrow \vec{v} \cdot \vec{v} = 0 \quad \text{If } \rho = \text{constant. Incompressible flow.}$$

Momentum flux:

$$\Pi_{\alpha\beta} = (\rho v_\alpha) v_\beta$$

Momentum density

$$P_\alpha = \rho v_\alpha$$

$$\Rightarrow \partial_t P_\alpha + \vec{\nabla} \cdot (\text{flux of } P_\alpha) = \partial_\beta \sigma_{\alpha\beta}$$

$$\Rightarrow \partial_t (\rho v_\alpha) + \vec{\nabla} \cdot \Pi_{\alpha\beta} = \partial_\beta \sigma_{\alpha\beta}$$

$$\Rightarrow \partial_t (\rho v_\alpha) + \partial_\beta (\rho v_\alpha v_\beta) = \partial_\beta \sigma_{\alpha\beta}$$

$$\Rightarrow \boxed{\partial_t (\rho v_\alpha) + \partial_\beta (\rho v_\alpha v_\beta - \sigma_{\alpha\beta}) = 0}$$

Reduces to the EULER eqn for
and

local
conservation of
momentum.

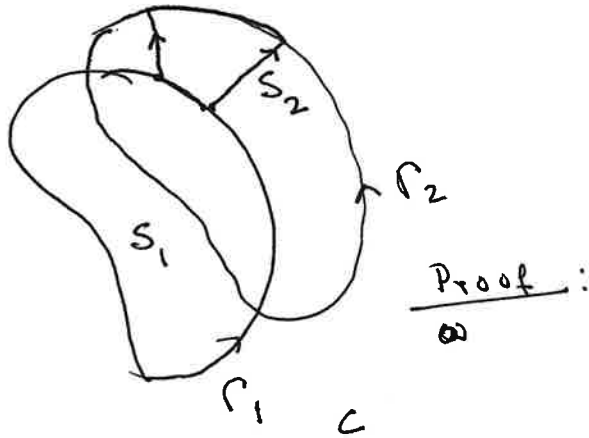
$$\sigma_{\alpha\beta} = -p \delta_{\alpha\beta}$$

$p = \text{constant}$

not necessarily closed

(4)

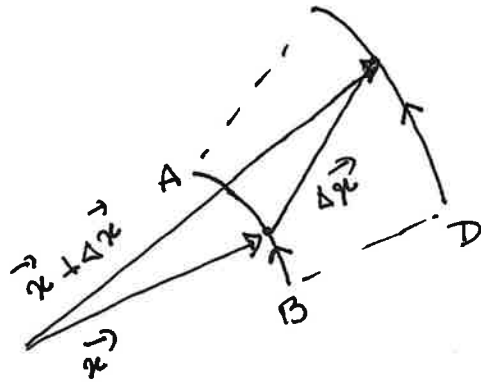
Kelvin's Theorem: The flux of vorticity through a Lagrangian surface remains constant in time in dry water.



$$\int_{S_1} \vec{\omega} \cdot \hat{n} dS = \int_{S_2} \vec{\omega} \cdot \hat{n} dS, \quad \vec{\omega} = \nabla \times \vec{u}$$

$$\int_S \vec{\omega} \cdot \hat{n} dS = \int_S (\nabla \times \vec{u}) \cdot \hat{n} dS = \oint_C \vec{u} \cdot d\vec{l}$$

$$\vec{BA} = d\vec{l} \quad \vec{DC} = d\vec{l}' \quad \Delta \vec{x} = \vec{u}(\vec{x}) \Delta t$$



$$\vec{u}(\vec{x} + \Delta \vec{x}, t + \Delta t) \cdot d\vec{l}'$$

$$= \vec{u}(\vec{x} + \Delta \vec{x}, t + \Delta t) \cdot d\vec{l}$$

$$= \left[\vec{u}(\vec{x}, t) + \frac{\partial \vec{u}}{\partial t} \Delta t + \frac{\partial \vec{u}}{\partial x_p} \Delta x_p + \text{h.o.t.} \right] d\vec{l}'$$

$$= \left[\vec{u} + \left(\frac{\partial \vec{u}}{\partial t} + u_p \frac{\partial \vec{u}}{\partial x_p} \right) \Delta t + \text{h.o.t.} \right] d\vec{l}'$$

5

$$A: \vec{r} + \frac{\partial \vec{y}}{\partial t} \quad B: \vec{r} - \frac{\partial \vec{y}}{\partial t} \quad D: \vec{r} - \frac{\partial \vec{y}}{\partial t} + \vec{v} \left(\vec{r} - \frac{\partial \vec{y}}{\partial t} \right) \Delta t$$

$$C: \vec{r} + \frac{\partial \vec{y}}{\partial t} + \vec{v} \left(\vec{r} + \frac{\partial \vec{y}}{\partial t} \right) \Delta t$$

$$\Rightarrow \vec{B}^A = \frac{d\vec{y}}{dt} = \vec{r} + \frac{\partial \vec{y}}{\partial t} = \vec{y} - \vec{r} + \frac{\partial \vec{y}}{\partial t} = \vec{y} - \vec{r} + \frac{\partial \vec{y}}{\partial t} \Delta t - \vec{r} + \frac{\partial \vec{y}}{\partial t} \Delta t$$

$$\vec{r}^C = \frac{d\vec{y}}{dt} = \vec{r} + \frac{\partial \vec{y}}{\partial t} + \vec{v} \left(\vec{r} + \frac{\partial \vec{y}}{\partial t} \right) \Delta t - \vec{r} + \frac{\partial \vec{y}}{\partial t} \Delta t - \left[\vec{v} \left(\vec{r} \right) - \frac{\partial \vec{v}}{\partial x} \frac{\partial \vec{y}}{\partial t} \right] \Delta t$$

$$= \frac{d\vec{y}}{dt} + \left[\vec{v} \left(\vec{r} \right) + \frac{\partial \vec{v}}{\partial x} \frac{\partial \vec{y}}{\partial t} \right] \Delta t + h.o.t.$$

$$= \frac{d\vec{y}}{dt} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{y} \Delta t + h.o.t.$$

$$\Rightarrow \vec{V} (t + \Delta t) \cdot \frac{d\vec{y}}{dt}$$

$$= \left[\vec{v} + \left(\frac{\partial \vec{v}}{\partial x} \vec{v} + \vec{v} \frac{\partial \vec{v}}{\partial x} \right) \Delta t \right] \left[\vec{y} + \left(\frac{\partial \vec{y}}{\partial t} \cdot \vec{v} \right) \vec{v} \Delta t \right]$$

$$= \vec{v} \cdot \frac{d\vec{y}}{dt} + \vec{v} \cdot \left(\frac{\partial \vec{y}}{\partial t} \cdot \vec{v} \right) \vec{v} + \left(\frac{\partial \vec{v}}{\partial x} \vec{v} + \vec{v} \frac{\partial \vec{v}}{\partial x} \right) \cdot \frac{d\vec{y}}{dt} \Delta t + h.o.t.$$

(6)

$$= \vec{v} \cdot d\vec{l} + \left(\partial_t + \vec{v} \cdot \vec{\nabla} \right) \vec{v} \cdot d\vec{l} \Delta t + \vec{\nabla} \left(\frac{v^2}{2} \right) \cdot d\vec{l} \Delta t.$$

Note $\oint_{\Gamma} \vec{\nabla} \left(\frac{v^2}{2} \right) \cdot d\vec{l} = 0$ ~~Stokes~~

$$\left(\partial_t + \vec{v} \cdot \vec{\nabla} \right) \vec{v} = - \vec{\nabla} \left(\frac{p}{\rho} + \Psi \right)$$

$$\Rightarrow \oint_{\Gamma} \left[\partial_t + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right] \cdot d\vec{l} = 0$$

$$\Rightarrow \oint_{\Gamma'} \vec{v}(\vec{x} + \Delta \vec{x}, t + \Delta t) \cdot d\vec{l}' = \oint_{\Gamma} \vec{v}(\vec{x}, t) \cdot d\vec{l}$$

Note 1: works for any external force which can be written as gradient of a potential.

EULER eqn.

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{\rho} \nabla p + \vec{g}$$

Bernoulli's theorem.

$$\frac{p}{\rho} + \frac{v^2}{2} = \text{constant on streamlines}$$

A local conservation law for a density & its flux.

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{v}) = 0$$

continuity eqn.

$$0 = \frac{\partial}{\partial t}(\rho \vec{v}) + \text{div}(\rho \vec{v} \otimes \vec{v} - \sigma \otimes \vec{x})$$

where Γ is a Lagrangian close contour.

$$\oint_{\Gamma} \vec{\omega} \cdot d\vec{l} = \text{constant}$$

Landau chapter 12 & 13 (Ignore * section)

lecture notes chapter 12 & 13

Summary:
Hydrostatics:
Dry water hydrodynamics: