

Boundary layer theory

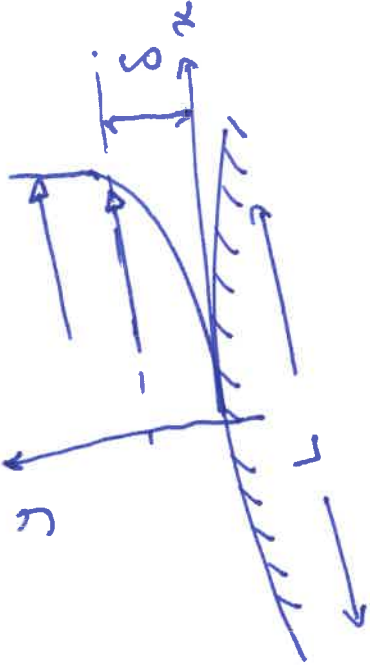
①

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} = \nu \nabla^2 \vec{v} - \nabla p / \rho$$

1. Steady (laminar) problem $\partial_t \vec{v} = 0$

2. Expt. input: velocity jumps close to its "free-stream" value within boundary layer δ

3. The boundary layer decreases as Re increases.



(2)

OUTSIDE

Gradients are small
viscosity is small

Ignore viscosity

Bernoulli's integral holds: $p + \frac{\rho}{2} v^2 = \text{constant}$ (no gravity)

$$\Rightarrow \partial_x p + v \partial_x v_x = 0$$

$$\Rightarrow \partial_x p = -\rho \partial_x U \quad \leftarrow \text{on a streamline } \vec{v} \sim U$$

BOUNDARY LAYER

$$(v_x \partial_x + v_y \partial_y) v_x = v (\partial_x^2 + \partial_y^2) v_x - \partial_x p$$

↓ small ↓ large
↑ small ↑ large
↑ small ↑ large

match with outer layer

systematically separate small from large $O(1)$

code: $O(1) \Rightarrow$ large; $Re = \frac{UL}{\nu} \sim \frac{UL}{\nu} \rightarrow \text{very large}$

small

③

$$v_x = v_0 v_1 ; \quad v_y = \left(\frac{\delta}{L}\right) v_0 v_2 = \varepsilon v_0 v_2$$

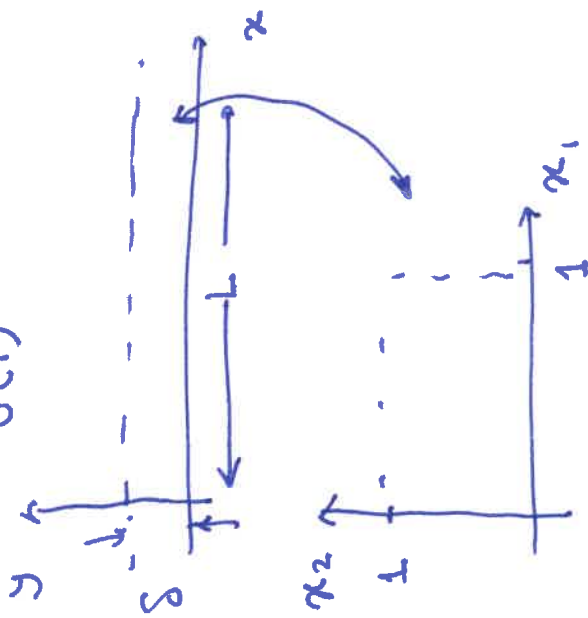
$\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$
 $O(1)$ $O(1)$ $O(1)$ $O(1)$
small small small small

$$x = \frac{1}{L} x_1 \quad y = L \left(\frac{\delta}{L}\right) x_2 = L \varepsilon x_2$$

$\uparrow$ $\uparrow$ $\uparrow$ $\uparrow$
 $O(1)$ $O(1)$ $O(1)$ $O(1)$
small small small small

$$\frac{\partial}{\partial x} = \frac{\partial x_1}{\partial x} \frac{\partial}{\partial x_1} = \frac{1}{L} \frac{\partial}{\partial x_1}$$

$$\frac{\partial}{\partial y} = \frac{\partial x_2}{\partial y} \frac{\partial}{\partial x_2} = \frac{1}{\varepsilon L} \frac{\partial}{\partial x_2}$$



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$$\left(\frac{U_0^2}{L}\right) (v_1 \partial_1 + \frac{\epsilon}{L} v_2 \partial_2) v_1 = \frac{U_0^2}{L^2} \partial_1^2 v_1 + \frac{U_0}{\epsilon^2 L^2} \partial_2^2 v_2 - \frac{U_0^2}{L} v_1 \partial_1 v_1'$$

$$\Rightarrow (v_1 \partial_1 + v_2 \partial_2) v_1 = \frac{1}{Re} \partial_1^2 v_1 + \frac{1}{\epsilon^2 Re} \partial_2^2 v_2 - v_1 \partial_1 v_1' ; Re = \frac{U_0 L}{\nu}$$

As $\nu \rightarrow 0, Re \rightarrow \infty ; \epsilon \rightarrow 0$ what about $\epsilon^2 Re$?

$$\epsilon^2 Re \rightarrow 1 \Rightarrow \epsilon = \frac{1}{\sqrt{Re}}$$

$$\boxed{\delta \sim \frac{L \nu}{U_0}}$$

Demand :

$$(v_1 \partial_1 + v_2 \partial_2) v_1 = \epsilon^2 \partial_1^2 v_1 + \partial_2^2 v_2 - v_1 \partial_1 v_1'$$

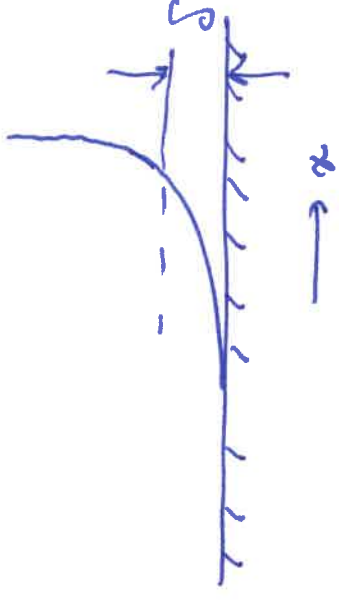
$$\Rightarrow (v_1 \partial_1 + v_2 \partial_2) v_1 = \partial_2^2 v_1 - v_1 \partial_1 v_1'$$

Eqn. for
laminar
boundary
layer

Application

Infinite flat plate

$$\delta \sim \sqrt{\frac{\nu x}{U_0}}$$



Beyond boundary layer :

As Re increases :

Boundary layer separates

Sometimes reattaches

Becomes unstable

Becomes TURBULENT