

Boundary layer theory

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Example

$$\varepsilon y'' + (1 + \varepsilon) y' + y = 0$$

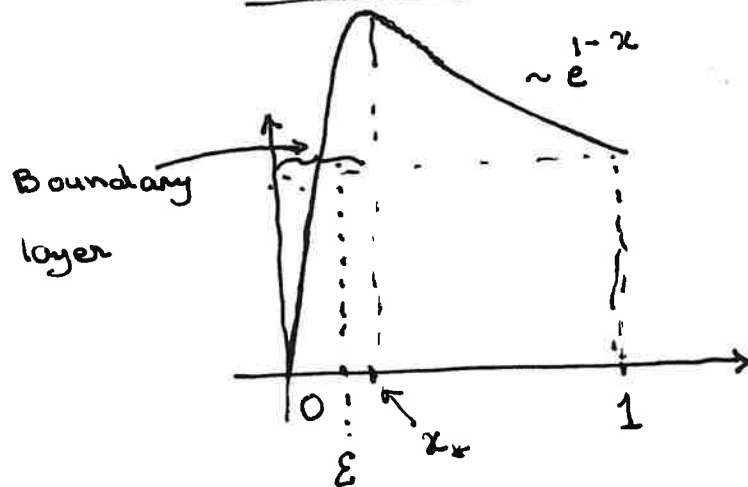
$$y(0) = 0, \quad y(1) = 1$$

Boundary layer near $x=0$
as $\varepsilon \rightarrow 0^+$

solution :

$$y(x) =$$

$$\frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}$$



For choose a fixed ε (small)

$$\text{For } x \gg \varepsilon, \quad \frac{x}{\varepsilon} \gg 1$$

$$\Rightarrow y(x) \approx \frac{e^{-x}}{e^{-1}} = e^{1-x}$$

$$\text{For } x \ll \varepsilon, \quad \frac{x}{\varepsilon} \ll 1$$

$$\Rightarrow y(x) \approx \left(\frac{1}{e^{-1} - e^{-1/\varepsilon}} \right) \left[(1 - x + \dots) - \left(1 - \frac{x}{\varepsilon} + \dots \right) \right]$$

$$\approx e^{1/\varepsilon} \left(\frac{1}{\varepsilon} - 1 \right) x + \dots$$

maximum at $x = x_*$

$$\left. \frac{dy}{dx} \right|_{x=x_*} = 0 \Rightarrow \left(-e^{-x_*} + \frac{1}{\varepsilon} e^{-x_*/\varepsilon} \right) = 0$$

$$e^{-x_*} = \frac{1}{\varepsilon} e^{-x_*/\varepsilon}$$

$\Rightarrow$

$$-x_* = \ln\left(\frac{1}{\varepsilon}\right) - \frac{x_*}{\varepsilon}$$

$\Rightarrow$

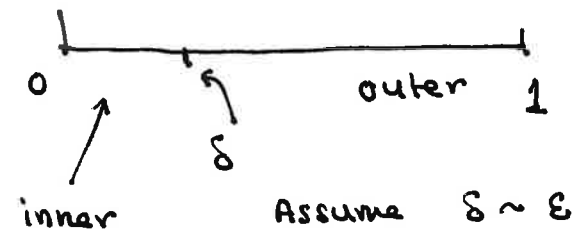
$$\Rightarrow x_* = \frac{\ln(1/\varepsilon)}{\left(\frac{1}{\varepsilon} - 1\right)} \approx \varepsilon \ln\left(\frac{1}{\varepsilon}\right) \leftarrow \text{small factor of } \varepsilon$$

How to solve it perturbatively?

x The solution is a function of x and ϵ

x There is an inner and outer limit-

$$(x \ll \delta) \quad (\delta \ll x \leq 1)$$



x Outer limit of the ODE: $y'_{out} + y_{out} = 0$
 solution: $y_{out} = A e^{-x} \Rightarrow y_{out} = e^{1-x}$

can fit the boundary condition at $x=1$, but not at $x=0$ too.

→ singular perturbation theory

x Inner limit: change of variable $x = \epsilon X$ ← varies ϵ times slower than x
 $y \rightarrow Y$

$$\frac{d^2 Y}{dX^2} + \frac{dY}{dX} = 0$$

$$\Rightarrow Y(X) = A - e^{B-X} \quad \text{at } x=0, Y=0 \Rightarrow A = e^B$$

$$\Rightarrow Y(X) = e^B (1 - e^{-X})$$

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Together

$$Y_{in}(X) = e^B (1 - e^{-X}) , \quad Y_{out}(x) = e^{1-x}$$

x Asymptotic matching : intermediate region limit : $\left. \begin{array}{l} x \rightarrow 0 \\ \varepsilon \rightarrow 0^+ \\ X = \frac{x}{\varepsilon} \rightarrow \infty \end{array} \right\}$

$$\lim_{X \rightarrow \infty} Y_{in}(X) = e^B$$

$$\lim_{X \rightarrow \infty} Y_{in}(X) = \lim_{x \rightarrow 0} Y_{out}(x)$$

$$\lim_{x \rightarrow 0} Y_{out}(x) = e$$

$$\Rightarrow B = 1$$

The intermediate limit is not always a number

$$y_{in}(x) = e (1 - e^{-x/\varepsilon})$$

$$Y_{out}(x) = e^{1-x}$$

x Does the overlap region always exist?

Assume it does and then check that it actually does.

x Is $\delta \sim \varepsilon$? Not necessarily. In other problems $\delta \sim \varepsilon^p$