

DHRUBADITYA MITRA

FIRST INTRODUCTION TO SOFT MATTER

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1 Introduction and overview

1.1 What are soft matter ?

As the name suggests soft-matter are material that are soft to touch. It could be a wide range of material, soap, mayonnaise, jam, ketchup, tooth-paste, including of course the skin of our own hands. The general property of all these is that they are *complex*. For example, they may not be either solid or liquid but something in between. A particularly interesting soft-matter which is not soft is glass. Another soft material that has had immense impact on technology and also our fundamental understanding of nature are the liquid crystals. Practically any biological material, e.g., from a piece of cotton cloth to a single red blood cell is soft material. One way to classify them is in contrast with *hard matter*. The name hard matter is typically applied to solids that form regular crystalline structures, including defects. To this long list of things I also add simple fluids, like water.

As physicists we are not too fond of making of long list of materials and calling these objects of interest. We leave such approach to stamp collectors. We rather look for their unifying characteristics. The following are certain features that distinguishes soft material.

- They are typically held together by covalent bonds, not ionic bonds. Sometime they are held together by even weaker forces, e.g., Van der Waals forces. These implies that their elastic coefficients, for example, Young's modulus, is smaller, which is really what defines them as soft. Although for many of them defining a elastic modulus itself may require care.
- They often involve material that are much larger than atomic scales but at the scale of macro-molecules. For example, polymers or colloids. In other words – they often live in meso scales.
- Fluctuations play crucial roles, these fluctuations may be thermal but they also may be come from other sources for example from the chaotic behavior of the material themselves.

I often define soft-matter physics more simply, whatever gets

published in Physics Review E belongs in soft-matter, thereby leaving the job of defining soft-matter to the editors of PRE.

1.2 *Why soft-matter ?*

One answer to this question is obvious, if almost all material we use in our everyday life is soft then this must be the part of physics with widest possible application. But, you may complain, that this seems to be encroaching into a domain that belongs to chemists. If you want a certain material with certain elastic properties or certain viscosity you go to a chemist and ask them to do their magic of mixing stuff with different colors. Or to study a red blood cell you go to a biologist. What is the role of physics in this field or formed in different way, what kind of questions does a soft-matter physicist asks and how does she goes about answering them ? This is the purpose of this course, I will return to this question at the end of the course. So I will give a short and incomplete answer at this moment. I will give you a few examples of the kind of physics questions in this field.

- How soft is a red blood cell ? You may ask why should I care about how soft a blood cells is, does it determine its biological behavior ? It turns out, it does. The mechanics properties of biological material play crucial role in their functions.
- If you add a few long chained polymers in a turbulent pipe flow, for example a fire hose, you will find that the drag of flow has significantly reduced. The same pressure head that was earlier driving the flow up to second floor now reaches the fourth floor. How do we understand this phenomena ?
- Consider collective motion of a large number of self-propelled organisms, for example a herd of wild beasts or a school of fish. Each individual can see only their neighbors and follow them while the whole collective moves in certain directions. Can we write an equation for the velocity and density field of such a collective ? When such a collective goes around an obstacle, for example around a large stone does it flow like water ?
- Consider a collection of people. Each has an opinion about a certain question, for example, each of them may have a different opinion about the effectivity of vaccination against covid 19. Each one of them may trust somebody else and also mistrust some other people. What kind of equations describe the collective dynamics of such opinion ?



Marie-Sophie Germain, 1776 to 1831, was a French mathematician who made pioneering contributions to theory of elasticity. She came from a wealthy family (her father was formally a bourgeoisie). She was a teenager at the time of the French revolution had very little formal education in mathematics as was usual for women at those times in Europe. She learnt from correspondence with Lagrange, Legendre and Gauss. Her work on elasticity was one of the first attempts to write down the equations for vibrations of a plate. She was the first woman to win the grand prize of the Paris academy of sciences for this work. Contemporary mathematicians considered her work to suffer from an absence of rigor, so possibly she was more a physicist than a mathematician in this field. She also made important contributions to number theory.

1.3 Tools of soft-matter

As this is a theoretical course my goal is not to cover all corners of soft matter physics. My goal is to give you a framework, as general as possible, that will help you understanding the physics of soft matter. I will try to cover well a set of basic tools that are necessary. But even that will be incomplete.

1.3.1 Continuum mechanics

The tools that we will need started being developed quite some time back around the time of Euler. From the time of Euler to the first half of 20th century these field was called continuum mechanics – elasticity and fluid mechanics. Every physicist had some background in these tools. But starting from the second half of 20th century quantum mechanics was brought in and to make space (rather time) in the curriculum continuum mechanics of moved out to the engineering departments. Now the same topics are back but in graduate curriculum and taught with somewhat different viewpoint, you could call these neo-classical physics if you like.

1.3.2 Equilibrium and non-equilibrium

The second concept crucial to understanding soft matter systems is statistical mechanics or thermodynamics. As soft-matter systems contains a large number of basic constituents it is necessary to take a statistical approach. But an additional difficulty is that these systems are often not in equilibrium. The equilibrium statistical mechanics that you have learnt so far will not work. In this set of lectures we will not have a scope to develop the formalism necessary to deal with such problems in general. I will show you, in brief, how the equations of continuum mechanics appear at large scales if we assume local equilibrium at small scales.

1.3.3 Phases

The other concept that plays a crucial role in soft-matter is the concept of phases. From thermodynamics you are familiar with three phases, solid, liquids and gases. But there may be more phases. For example you can have a collection of rod like molecules whose centers are not on a regular lattice so they do not form a solid but they all point towards a particular direction so there is some directional order in the system. Such a system is a nematic. The physics of transition between phases is a subject in itself – **phase transition**. The theory of phase transition is one of the most profound and beautiful



Arnold Johannes Wilhelm Sommerfeld, 1868 to 1951, was a German theoretical physicist whose pioneering contributions to early atomic and quantum physics is well-known. Sommerfeld was also an educator per-excellence – his lectures on theoretical physics, which are still in print, should be read by every serious student. His contributions to fluid mechanics is less well-known but no less important. He is the father of lubrication theory, particularly its application to understand wet friction. The Orr-Sommerfeld equation is a cornerstone in studies of instabilities in shear flows. Three of his doctoral students, Ludwig Hopf – Hopf bifurcation, Fritz Noether – physicists are more familiar with his sister Emmy – and Werner Heisenberg made crucial contribution to fluid mechanics. For further details see .

Michael Eckert. Fluid mechanics in sommerfeld's school. *Annual Review of Fluid Mechanics*, 47:1–20, 2015

theories of physics. The problem gets even more complex once you consider systems outside equilibrium. You can think of a collection of wild beasts whose position are not ordered but their direction of motion is. This is also a phase but not one that can be described within the ambit of equilibrium statistical physics.

1.3.4 Symmetry and order

Different phases are typically characterized by their symmetries. For example compare a solid and a fluid (liquid or gas). The molecules in a solid are arranged in a regular array while the position of molecules in a fluid are random. Hence fluid is both homogeneous and isotropic (statistically) but a solid is not. Let me try to be more precise. To give a theoretical description of any system I have to choose a coordinate system. In a fluid, if I translate my coordinate system by any amount the equations that describe the fluid must remain the same. Similarly, if I rotate my coordinate system by any amount the equations that describe the fluid must remain the same. But not so in a solid. In a solid the equations remain the same only if I shift my coordinate system by a lattice spacing or rotate it by a symmetry angle of the lattice. Thus a solid has *less symmetry than a fluid and more order*. Liquid crystals will lie somewhere in between. Some of their phases have the same symmetries as fluids. But they may also have phases where all the rod like molecules point in a specific direction while their centers are randomly distributed – more ordered than a fluid but less ordered than a solid.

One way to quantify order is by the radial distribution function $g(r)$. Take the origin of your coordinate system at one molecule. Count the number of molecules whose center lies within a sphere of radius r . Let us call this number $n(r)$. Then the number of molecules whose center are within a shell between r to $r + dr$ is dn/dr . The radial distribution function is defined as

$$g(r) \equiv \frac{1}{4\pi r^2} \frac{dn}{dr} \quad (1.1)$$

For a homogeneous fluid the number of molecules that lie within a sphere of radius r is proportional to r^3 , hence

$$g(r) \sim \text{constant} \quad (1.2)$$

Of course, we do not expect this result to be valid when r is very small comparable to the intermolecular distance. We also do not expect this result to be true when r is very large, e.g., the size of the container that holds the fluid. With the understanding that we first take the limit of the size of the container going to infinity

$$g(r) \sim \text{constant} \quad \text{for } r \rightarrow \infty \quad (1.3)$$

In a solid you expect the $g(r)$ to have definite peaks at the distances that corresponds to integer multiples of lattice spacings.

1.3.5 Fractals

Let us now consider a more weird object a random fractal. A typical example is a long polymer. In a model of polymer – a model that does manage to explain several aspects of polymer physics – we represent the polymer as a random walk with a fixed step size. At every node of this random walk sits a monomer molecule. In principle, such a random walk cannot cross itself – it must be *self-avoiding*. Let us ignore this aspect for the moment. Then you know that the average distance covered by the random walker is zero, $\langle R \rangle = 0$ but the mean-square distance is proportional to the number of steps $\langle R^2 \rangle \sim N$. Applied to the polymer it means that the step size must be replaced by the number of monomers. A Brownian walk is scale-invariant, it looks the same at all scales ¹. Thus the number of monomers within a distance r from a single monomer is

$$n(r) \propto r^2 \quad (1.4)$$

Hence the radial distribution function will be

$$g(r) \sim \frac{1}{r} \quad (1.5)$$

¹Of course, a Brownian walk with finite number of step (however large) with a fixed step-size is scale invariant only for scales larger than the step size and smaller than the maximum distance covered. Such provisos will often appear in our arguments. Henceforth, I will assume that you can add such provisos yourself and drop them.

1.3.6 Linear and non-linear

The other concept which will appear several times in this course is nonlinearity. Natural world are nonlinear whereas most of the tools we have developed in physics are able to deal with linear problems. Even the gravitational forces between two bodies is nonlinear. This is why we do not know how to solve for gravitational problems with three or more bodies. The theory of electromagnetism is a linear theory – the Maxwell's equations are linear – in vacuum. The equations that describe the flow of fluids, the Navier–Stokes equations are nonlinear. For materials like tooth-paste the equations of elasticity in three dimensions is nonlinear. The elastic membrane of a drum obeys close to linear physics if the drum is not hit too hard. But for large vibrations the elasticity of beams and plates is a strongly nonlinear problem.

1.3.7 Dimensional analysis

Finally, let me introduce a very useful but under appreciated tool, *dimensional analysis*. Before we dive into it, consider small and large. In the units we are familiar with a certain quantity may sound quite

large or quite small. For example, the radius of earth is about 6400 km which sounds quite large. Which just means the radius is very large compared to the typical size of humans, which is the typical length scale used in meter. But compared to the distance Sun to Earth, which is about 150×10^6 km the radius of earth seems quite small. The lesson being small or large should always be thought in terms of *dimensionless* numbers.

To the best of my knowledge the idea that every term in an equation should have the same dimension was first clearly stated by Fourier.

The fundamental theorem of dimensional analysis, often called Buckingham's pi theorem is in any physical problem involving a number of dimensional quantities, the relationship between them can be expressed by forming all possible independent dimensionless quantities, denoted by Π_1, Π_2 etc. Then the solution to the physical problem is for the form

$$\Pi = f(\Pi_1, \Pi_2, \dots) \quad (1.6)$$

where f is a function of n variables. Sometimes there is only one possible dimensionless combination. Then the solution is fixed up to a constant. This simple idea can be very powerful. For example, let us try to calculate the drag force (F) on a sphere in a fluid. What are the physical quantities responsible. One the velocity of the sphere (v). Two, the viscosity of the fluid (η) – dimension of viscosity is L^2/T . Three, the radius of the sphere. The only possible combination that gives a force can be ηav . It implies that the force must be the following expression

$$F = \eta av f(\text{Re}) \quad (1.7)$$

where $\text{Re} = av/\eta$.

1.4 Notes

The Wikipedia article on soft-matter (https://en.wikipedia.org/wiki/Soft_matter) is a good introduction to this subject. Although my definition of soft-matter is wider.

Some of the pioneers of modern soft matter are Sam Edwards, Leo Kadanoff, P-G De Gennes, Masao Doi and Francoise Brochard-Wyart. The Nobel lecture of De Gennes [[de Gennes](#)] is an excellent invitation.

The measurement of softness of red blood cells and of soft biological tissue in general is subfield of biophysics in itself. You can start by marveling at the encyclopedic physical biology of the cell [[Phillips et al., 2012](#)].

The problem of drag reduction by addition of polymers was first observed by Toms and by Mesels although it often goes by the name of Toms effect. A relatively modern summary of the state of this problem can be found in a Reviews of Modern Physics article by Procaccia, L'vov and Benzi [Procaccia et al., 2008].

The physics of a collection of self-propelled particles forms a cornerstone of modern statistical physics more generally called active matter. Two recent reviews [Marchetti et al., 2013, Ramaswamy, 2010] summarizes the present state of this field.

An excellent example of recent work on dynamics of opinions is an article by Joseph Baron in Physical Review E [Baron, 2021].

A collection of papers [PRE] to celebrate the 25 years of Physical Review E captures the wide variety of problems that fits within this framework.

Exercise 1

1. In self-avoiding random walk the mean-square-distance after N step is $\langle R^2 \rangle \sim N^{3/2}$. Calculate the pair distribution function $g(r)$.
2. In a magnetic material in its paramagnetic phase the spins of individual atoms can point in any random direction in three dimensions. As the temperature is lowered below the critical temperature the material goes into a ferromagnetic phase where all the spins point in the same direction. Which symmetry is being broken in this phase transition ?
3. In a liquid solid transition which symmetry is being broken ?

2 Strain

2.1 Helmholtz's theorem

Let us start with an experiment commonly performed by six year olds in Sweden, we take a block of “slime” and deform it. Before deforming we mark two point, A and B, on this slime with a different color. After the deformation these two points have moved to two new positions. The vector that connected these two points has also changed to a new vector. Following Sommerfeld, [Sommerfeld, 2006] I state the following:

Theorem. *The most general motion of a sufficiently small element of a deformable (i.e., not rigid) body can be represented as the sum of*

1. *a translation*
2. *a rotation*
3. *an extension (contraction) in three mutually orthogonal directions.*

The original proof of this theorem is due to Helmholtz. The proof is based on Taylor series expansion as we describe below.

Let us use a Cartesian coordinate system, in which the coordinates of a point A are given by x, y, z . Under deformation every point in the “slime” as moved to new position. Thus to every point in the dough I can associate a vector $\mathbf{u}$ which denotes the displacement of that point under this deformation. This vector $\mathbf{u}$ itself can then be considered as a function of the three space coordinates x, y, z . $\mathbf{u}(x, y, z)$ is then a *vector field*. You have encountered fields before, for example the temperature in this room is a *scalar field*. To imagine a scalar field think of a number at every point in space. To imagine a vector field think of an arrow associated with every point in space. A typical example of a vector field will be a gravitational field. Consider a spec of dirt. You take it to every point around the Earth and calculate the gravitational force the earth exerts on it. Then divide the force by the mass of the dirt to get force-per-unit-mass. This is a vector that at every point in space will point towards the center of the Earth (if Earth were a perfect sphere) and its magnitude will decrease



Arnold Johannes Wilhelm Sommerfeld, 1868 to 1951, was a German theoretical physicist whose pioneering contributions to early atomic and quantum physics is well-known. Sommerfeld was also an educator per-excellence – his lectures on theoretical physics, which are still in print, should be read by every serious student. His contributions to fluid mechanics is less well-known but no less important. He is the father of lubrication theory, particularly its application to understand wet friction. The Orr–Sommerfeld equation is a cornerstone in studies of instabilities in shear flows. Three of his doctoral students, Ludwig Hopf – Hopf bifurcation, Fritz Noether – physicists are more familiar with his sister Emmy – and Werner Heisenberg made crucial contribution to fluid mechanics. For further details see .

Michael Eckert. Fluid mechanics in sommerfeld's school. *Annual Review of Fluid Mechanics*, 47:1–20, 2015

as the spec of dirt is moved further and further away from Earth. Similarly we now consider another vector field, that of displacement of material points. Every point in the slime has been displaced: this displacement itself, in general, is a function of position given by the vector field $\mathbf{u}$. Then consider two neighboring points $A(x, y, z)$ and $B(x + \Delta x, y + \Delta y, z + \Delta z)$. Under deformation, A has moved to A' and B has moved to B' . The vector $\overrightarrow{AA'}$ is given by $\mathbf{u}(x, y, z)$ and the vector $\overrightarrow{BB'}$ is given by $\mathbf{u}(x + \Delta x, y + \Delta y, z + \Delta z)$. I can now expand each Cartesian component of the vector field $\mathbf{u}(x, y, z)$, given by $u_1(x, y, z)$, $u_2(x, y, z)$ and $u_3(x, y, z)$ in a Taylor series¹

$$u_1(x + \Delta x, y + \Delta y, z + \Delta z) = u_1(x, y, z) + \frac{\partial u_x}{\partial x} \Delta x + \frac{\partial u_x}{\partial y} \Delta y + \frac{\partial u_x}{\partial z} \Delta z + \dots \quad (2.1a)$$

$$u_2(x + \Delta x, y + \Delta y, z + \Delta z) = u_2(x, y, z) + \frac{\partial u_y}{\partial x} \Delta x + \frac{\partial u_y}{\partial y} \Delta y + \frac{\partial u_y}{\partial z} \Delta z + \dots \quad (2.1b)$$

$$u_3(x + \Delta x, y + \Delta y, z + \Delta z) = u_3(x, y, z) + \frac{\partial u_z}{\partial x} \Delta x + \frac{\partial u_z}{\partial y} \Delta y + \frac{\partial u_z}{\partial z} \Delta z + \dots \quad (2.1c)$$

Let us compactify our notation. I use u_α where $\alpha = 1, 2, 3$ for a component of the vector $\mathbf{u}$. Let $\mathbf{r} = (x, y, z)$ and $\Delta \mathbf{r} = (\Delta x, \Delta y, \Delta z)$. And I use the symbols

$$a_{11} = \frac{\partial u_x}{\partial x}, \quad a_{12} = \frac{\partial u_x}{\partial y}, \quad a_{13} = \frac{\partial u_x}{\partial z}, \quad (2.2a)$$

$$a_{21} = \frac{\partial u_y}{\partial x}, \quad a_{22} = \frac{\partial u_y}{\partial y}, \quad a_{23} = \frac{\partial u_y}{\partial z}, \quad (2.2b)$$

$$a_{31} = \frac{\partial u_z}{\partial x}, \quad a_{32} = \frac{\partial u_z}{\partial y}, \quad a_{33} = \frac{\partial u_z}{\partial z}, \quad (2.2c)$$

I can write the equation for deformation to be

$$u_\alpha(\mathbf{r} + \Delta \mathbf{r}) = u_\alpha(\mathbf{r}) + a_{\alpha\beta} \Delta r_\beta + \dots \quad (2.3)$$

Remember that the collection of number $a_{\alpha\beta}$ themselves are function of space, they depend on *where* we calculate the derivatives. As long as $|\Delta \mathbf{r}|$ is small enough I can ignore all the higher order terms that I have denoted by the triple-dots. This is assumed in the rest of this discussion.

Now I am going to reorganize the collection of numbers $a_{\alpha\beta}$ in “symmetric” and “anti-symmetric” manner:

$$a_{\alpha\beta} = \frac{a_{\alpha\beta} - a_{\beta\alpha}}{2} + \frac{a_{\alpha\beta} + a_{\beta\alpha}}{2}. \quad (2.4)$$

¹This assumes that the field of displacement is “smooth” enough that the derivatives exists. This need not always be the case, for example, if you deform the slime in a way that there is hole inside it then the displacement field may no longer be Taylor expandable. No such case will be considered in these lectures.

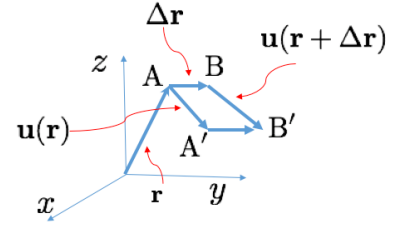


Figure 2.1: We use a Cartesian coordinate system. In this system, the point A has the position vector $\mathbf{r}$, and its neighboring point B has the position vector $\mathbf{r} + \Delta \mathbf{r}$. Under deformation A moves to A' and B moves to B' . The deformation at A is $\mathbf{u}(\mathbf{r})$ and the deformation at B is $\mathbf{u}(\mathbf{r} + \Delta \mathbf{r})$.

By $a_{\alpha\beta} \Delta r_\beta$ I mean $\sum_{\beta=1,2,3} a_{\alpha\beta} \Delta r_\beta$. In these lectures any index, e.g., β here, that is repeated is assumed to be summed – this is known as the Einstein summation convention.

Substituting back in Equation (2.1) I get

$$u_1(\mathbf{r} + \Delta\mathbf{r}) = u_1(\mathbf{r}) + 0 + \frac{a_{12} - a_{21}}{2} \Delta y + \frac{a_{13} - a_{31}}{2} \Delta z + a_{11} \Delta x + \frac{a_{12} + a_{21}}{2} \Delta y + \frac{a_{13} + a_{31}}{2} \Delta z \quad (2.5a)$$

$$u_2(\mathbf{r} + \Delta\mathbf{r}) = u_2(\mathbf{r}) + \frac{a_{21} - a_{12}}{2} \Delta x + 0 + \frac{a_{23} - a_{32}}{2} \Delta z + \frac{a_{12} + a_{21}}{2} \Delta x + a_{22} \Delta y + \frac{a_{23} + a_{32}}{2} \Delta z \quad (2.5b)$$

$$u_3(\mathbf{r} + \Delta\mathbf{r}) = u_3(\mathbf{r}) + \frac{a_{31} - a_{13}}{2} \Delta x + \frac{a_{32} - a_{23}}{2} \Delta y + 0 + \frac{a_{31} + a_{13}}{2} \Delta x + \frac{a_{23} + a_{32}}{2} \Delta y + a_{33} \Delta z \quad (2.5c)$$

I can now interpret the right-hand-side(RHS) of Equation (2.5) as a sum of three displacements,

$$\mathbf{u}(\mathbf{r} + \Delta\mathbf{r}) = \mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_3 \quad (2.6)$$

The black ones show that two neighboring points are displaced by exactly the same amount,

$$\mathbf{u}(\mathbf{r} + \Delta\mathbf{r}) = \mathbf{u}(\mathbf{r}), \quad (2.7)$$

this is translation.

Now organize the red ones

$$\mathbf{d}_2 = \begin{pmatrix} 0 & \frac{a_{12} - a_{21}}{2} & \frac{a_{13} - a_{31}}{2} \\ \frac{a_{21} - a_{12}}{2} & 0 & \frac{a_{23} - a_{32}}{2} \\ \frac{a_{31} - a_{13}}{2} & \frac{a_{32} - a_{23}}{2} & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \quad (2.8)$$

The matrix that appears is an anti-symmetric matrix.

A different but equivalent way to interpreting $\mathbf{d}_2$ is

$$\mathbf{d}_2 = \frac{1}{2} \Delta\boldsymbol{\phi} \times \Delta\mathbf{r} \quad (2.9)$$

where the vector $\Delta\boldsymbol{\phi} \equiv (\Delta\phi_1, \Delta\phi_2, \Delta\phi_3)$ is an infinitesimal rotation vector which points along the axis of rotation with a magnitude equal to the angle of rotation, with

$$\Delta\phi_1 = a_{32} - a_{23}, \quad (2.10a)$$

$$\Delta\phi_2 = a_{13} - a_{31}, \quad (2.10b)$$

$$\Delta\phi_3 = a_{21} - a_{12}. \quad (2.10c)$$

Let us now consider the blue terms in Equation (2.5)

$$\mathbf{d}_3 = \begin{pmatrix} a_{11} & \frac{a_{12} + a_{21}}{2} & \frac{a_{13} + a_{31}}{2} \\ \frac{a_{21} + a_{12}}{2} & a_{22} & \frac{a_{23} + a_{32}}{2} \\ \frac{a_{31} + a_{13}}{2} & \frac{a_{32} + a_{23}}{2} & a_{33} \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \quad (2.11)$$

This is a real symmetric matrix. Hence it can always be diagonalized by an orthogonal transformation. In other words, by a suitable rotation of coordinates I can always find a new coordinate system in

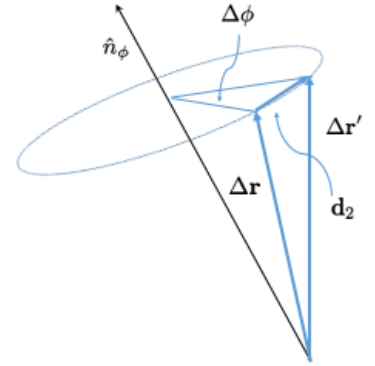


Figure 2.2: Infinitesimal rotation by cross-product. Write the vector $\Delta\boldsymbol{\phi}$ as a vector pointing in the direction $\hat{n}_\phi$ with magnitude $\Delta\phi$. The cross-product of $\Delta\boldsymbol{\phi}$ and $\Delta\mathbf{r}$ is the vector $\Delta\mathbf{r}'$.

An infinitesimal rotation can be represented by a vector, although not an "usual" vector but a *axial* vector. Rotation by a finite amount cannot in general be represented by a vector but can be represented by an anti-symmetric matrix as in Equation (2.8).

which this matrix is diagonal. In that coordinate the displacement $\mathbf{d}_3$ will have the form

$$\mathbf{d}_3 = \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{pmatrix} \begin{pmatrix} \Delta X_1 \\ \Delta X_2 \\ \Delta X_3 \end{pmatrix} \quad (2.12)$$

You may be already familiar with real symmetric matrices and their properties, in particular their eigenvalues and eigenvectors from a course on quantum mechanics where they appear as Hamiltonian or from a course in classical mechanics where they appear during the discussions of moment of inertia of rigid objects. Proofs can be found in any book on matrices or linear algebra, e.g., Arfken and Weber [Arfken and Weber, 1999].

The three Λ s are the three eigenvalues of the strain matrix $s_{\alpha\beta}$ given in Equation (2.11). They must be real, but they can be either positive or negative. A positive(negative) Λ denotes extension (compression) along the corresponding eigenvector $\mathbf{X}$. The three eigenvectors are orthogonal to each other, together they form a Cartesian triad. This proves Helmholtz's theorem.

2.2 Looking back

Let me point out a few crucial aspects of the proof we just presented.

1. I defined the displacement field $\mathbf{u}(\mathbf{x})$ as a function of the initial position every point $\mathbf{x}$. Instead I could define the displacement field as a function of the final position of the points. That would have also worked equally well.
2. The Helmholtz theorem is crucial in both elasticity and fluid mechanics. In elasticity, the strain plays the crucial role. In fluid mechanics, it is not strain but rate-of-strain that is important – we shall revisit them point. In addition, the rate-of-rotation appears as vorticity, a crucial quantity in fluid mechanics.
3. There is a subtle aspect of Helmholtz's theorem that needs some care. To see this let us revisit our derivation. Let the point A has the position vector $\mathbf{r}$ and the point B the position vector $\mathbf{r} + \Delta\mathbf{r}$. Then the position vector of the point A' is $\mathbf{r} + \mathbf{u}(\mathbf{r})$ and the coordinate of the point B' is $\mathbf{r} + \Delta\mathbf{r} + \mathbf{u}(\mathbf{r} + \Delta\mathbf{r})$. The vector $\overrightarrow{A'B'}$ is then

$$\Delta\mathbf{r}' = \mathbf{r} + \Delta\mathbf{r} + \mathbf{u}(\mathbf{r} + \Delta\mathbf{r}) - (\mathbf{r} + \mathbf{u}(\mathbf{r})) \quad (2.13)$$

$$= \Delta\mathbf{r} + \mathbf{u}(\mathbf{r} + \Delta\mathbf{r}) - \mathbf{u}(\mathbf{r}) \quad (2.14)$$

Then we used the approximation

$$\Delta\mathbf{u}(\mathbf{r}) \equiv \mathbf{u}(\mathbf{r} + \Delta\mathbf{r}) - \mathbf{u}(\mathbf{r}) = a_{\alpha\beta}\Delta r_\beta \quad (2.15)$$

Let us now calculate the square of the length of the vector $\Delta \mathbf{r}'$

$$\Delta \mathbf{r}' \cdot \Delta \mathbf{r}' \equiv \Delta r'_\mu \Delta r'_\mu \quad (2.16a)$$

$$= (\Delta r_\mu + a_{\mu\nu} \Delta r_\nu)(\Delta r_\mu + a_{\mu\beta} \Delta r_\beta) \quad (2.16b)$$

$$= \Delta r_\mu \Delta r_\mu + a_{\mu\nu} \Delta r_\mu \Delta r_\nu + a_{\mu\beta} \Delta r_\mu \Delta r_\beta + a_{\mu\nu} a_{\mu\beta} \Delta r_\nu \Delta r_\beta \quad (2.16c)$$

$$= \Delta r_\mu \Delta r_\mu + a_{\mu\nu} \Delta r_\mu \Delta r_\nu + \mathbf{a_{\mu\nu} \Delta r_\mu \Delta r_\nu} + a_{\mu\nu} a_{\mu\beta} \Delta r_\nu \Delta r_\beta \quad (2.16d)$$

$$= \Delta r_\mu \Delta r_\mu + (a_{\mu\nu} + \mathbf{a_{\nu\mu}}) \Delta r_\mu \Delta r_\nu + a_{\mu\nu} a_{\mu\beta} \Delta r_\nu \Delta r_\beta \quad (2.16e)$$

$$= \Delta r_\mu \Delta r_\mu + (a_{\beta\nu} + a_{\nu\beta}) \Delta r_\beta \Delta r_\nu + a_{\mu\nu} a_{\mu\beta} \Delta r_\nu \Delta r_\beta \quad (2.16f)$$

$$= \Delta r_\mu \Delta r_\mu + (a_{\beta\nu} + a_{\nu\beta} + a_{\mu\nu} a_{\mu\beta}) \Delta r_\beta \Delta r_\nu \quad (2.16g)$$

$$= \Delta r_\mu \Delta r_\mu + 2U_{\beta\nu} \Delta r_\beta \Delta r_\nu \quad (2.16h)$$

Here the last line is the definition of a matrix $U_{\beta\nu}$

$$U_{\alpha\beta} \equiv \frac{1}{2} (a_{\alpha\beta} + a_{\beta\alpha} + \mathbf{a_{\mu\alpha} a_{\mu\beta}}) \quad (2.17)$$

Notice that the matrix $U_{\alpha\beta}$ is the same as the strain matrix $s_{\alpha\beta}$ that we have defined before except the additional last term marked in red. The anti-symmetric part of $a_{\alpha\beta}$ has not appeared because rotation does not change the length of a vector. So far the only approximation we have made is keeping only the first term in the Taylor series expansion of $\mathbf{u}(\mathbf{r})$. If we now make the additional assumption that the terms in the gradient matrix itself are small, then the term $a_{\mu\alpha} a_{\mu\beta}$ are smaller and can be ignored. We shall revisit this assumption in the later chapters.

2.3 Imagining strain

The fact that the strain matrix can always be diagonalised to find three orthogonal eigenvectors helps in thinking about strain. To develop some geometric intuition let us consider the strain in its principal coordinate, we denote this axis by X_1, X_2, X_3 . Let us ignore the displacement $\mathbf{d}_2$ that signifies rotation.

2.3.1 Strain quadratic

One way to visualize the strain matrix is to construct what is called its “quadratic form”, $f(x, y, z) \equiv \mathbf{r}_\alpha s_{\alpha\beta} \mathbf{r}_\beta$ where the vector $\mathbf{r} =$

I assume this is your first exposure to the rule, repeated indices are summed, so I show all the intermediate steps to make the indices manipulation explicit. The expression

$$a_{\mu\nu} \Delta r_\mu \Delta r_\nu = \sum_{m,n} a_{\mu\nu} \Delta r_\mu \Delta r_\nu$$

The indices that are being summed over are called dummy indices. Obviously they can be denoted by any other symbol, i.e.,

$$a_{\mu\nu} \Delta r_\mu \Delta r_\nu = a_{\alpha\beta} \Delta r_\alpha \Delta r_\beta$$

I have not assumed that the matrix $a_{\mu\nu}$ is symmetric because it is not.

(x, y, z) has components r_α with $\alpha = 1, 2, 3$. Written explicitly as

$$f(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} a_{11} & \frac{a_{12}+a_{21}}{2} & \frac{a_{13}+a_{31}}{2} \\ \frac{a_{21}+a_{12}}{2} & a_{22} & \frac{a_{23}+a_{32}}{2} \\ \frac{a_{31}+a_{13}}{2} & \frac{a_{32}+a_{23}}{2} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (2.18)$$

The function $f(x, y, z)$ is in general a quadratic in all its arguments. Setting it equal to unity (or any other constant) defines a surface in three dimensional space. This surface is called a strain quadratic. It is an ellipsoid with three principal axis along X_1 , X_2 and X_3 if all the three eigenvalues of the strain matrix are positive.

2.3.2 Change in volume

Then consider a small volume element $dV = dX_1 dX_2 dX_3$. Under the deformation they change to

$$dV' = dX'_1 dX'_2 dX'_3 \quad (2.19a)$$

$$= (1 + \Lambda_1) dX_1 (1 + \Lambda_2) dX_2 (1 + \Lambda_3) dX_3 \quad (2.19b)$$

$$= dV(1 + \Lambda_1 + \Lambda_2 + \Lambda_3 + \text{h.o.t}) \quad (2.19c)$$

where in the last step we have moved all terms quadratic or higher in Λ s in to h.o.t. As the strain itself is small, it is safe to ignore them. What (2.19c) shows is that (up to leading order) the change in volume is given by the sum of the eigenvalues of the strain matrix, or its trace. Although we have proved this result in the principal axis it is true in all other axis too because the trace of a matrix is invariant under rotation of coordinate axis.

2.3.3 Change in angle

How does the angle between two vectors change under deformation. We know that rotation and translation does not change the angle, so we consider a case with zero rotation and translation. Let two vector $\mathbf{a}$ and $\mathbf{b}$ change to $\mathbf{a}'$ and $\mathbf{b}'$ respectively. Then

$$\mathbf{a}'_\alpha \mathbf{b}'_\alpha = (a_\alpha + s_{\alpha\beta} a_\beta)(b_\alpha + s_{\alpha\gamma} b_\gamma) \quad (2.20a)$$

$$= a_\alpha b_\alpha + b_\alpha s_{\alpha\beta} a_\beta + a_\alpha s_{\alpha\gamma} b_\gamma + \text{h.o.t} \quad (2.20b)$$

$$= a_\alpha b_\alpha + 2b_\alpha s_{\alpha\beta} a_\beta \quad (2.20c)$$

If we now consider $\mathbf{a}$ and $\mathbf{b}$ to be unit vectors, then the change in the dot produce gives

$$\cos(\theta + \Delta\theta) - \cos(\theta) = 2b_\alpha s_{\alpha\beta} a_\beta \quad (2.21)$$

For small $\Delta\theta$ is reduces to

$$(\sin \theta) \Delta\theta = 2b_\alpha s_{\alpha\beta} a_\beta \quad (2.22)$$

Now consider $\mathbf{a}$ and $\mathbf{b}$ to be $\hat{x}$ and $\hat{y}$ respectively, then $\theta = \pi/2$, $\sin(\theta) = 1$, and $\Delta\theta = 2s_{12}$. The off-diagonal elements of the strain matrix gives the change in angle between the coordinate axis under the strain deformation. Note that only in the principal axis frame the off-diagonal elements are zero and the angle between the principal axis remains unchanged under strain as they should.

3 Strain is a tensor

3.1 Introduction

So far we interpreted the strain as a 3×3 matrix at every point in space. It is also viewed as a second rank tensor field. Any old collection of nine numbers at every point in space does not make a tensor field, just like any collection of three numbers does not make a vector field. They must satisfy the correct *transformation laws*. Strain is an excellent way to introduce tensors, which we proceed to do now.

A scalar field, which a single number at every point in space must be invariant under coordinate transformation. Any vector quantity does not remain invariant, but must change exactly like (remain *covariant*) distance between two points under coordinate transformation. A vector is a tensor of rank one, the strain is a tensor of rank two. The vorticity can be either thought of as a tensor of rank one, i.e., a vector but it is a *polar* vector, one that changes sign under the coordinate transformation $x \rightarrow -x$, $y \rightarrow -y$ and $z \rightarrow -z$. Or it can be interpreted as a second rank anti-symmetric tensor. In this course I assume you know what a vector is, but not what a second or higher rank tensor is – second or higher rank tensor are common occurrence in soft matter, I enthusiastically recommend a wonderful book by Rutherford Aris¹ for a comprehensive introduction to this subject. During this course we shall learn tensors as and when we need them, starting now.

Consider a coordinate system $\{x_1, x_2, x_3\}$. We do a linear transformation and obtain a new coordinate system $\{y_1, y_2, y_3\}$. The linear transformation that relates them is given by

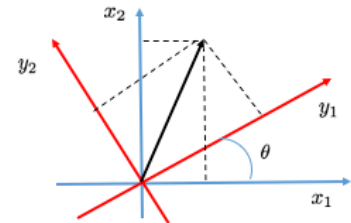
$$y_\alpha = a_{\alpha\beta} x_\beta, \quad (3.2)$$

where α, β runs from 1 to 3 and we have again assumed the Einstein summation convention. A typical example is rotation about an axis, as shown in Equation (3.1). The magnitude of the position vector remains unchanged under rotation; hence

$$y_\alpha y_\alpha = a_{\alpha\beta} a_{\alpha\mu} x_\beta x_\mu = x_\mu x_\mu. \quad (3.3)$$

Here, by coordinate transformation I mean only rotation of coordinates. The transformation properties under more general, nonlinear coordinate transformation gives rise to more general non-Cartesian tensors – they become useful while studying the general theory of relativity.

¹Rutherford Aris. Vectors, tensors and the basic equations of fluid mechanics. Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1962



An example of linear transformation of coordinates. The two Cartesian coordinate systems are related to each other by a rotation by the angle θ along an axis perpendicular to the plane. The same point in space are now labeled by two different coordinate systems (x_1, x_2) , and (y_1, y_2) related to each other by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (3.1)$$

Then we must have

$$a_{\alpha\beta} a_{\alpha\mu} = \delta_{\beta\mu} \quad (3.4)$$

where $\delta_{\beta\mu}$

$$\delta_{\beta\mu} = \begin{cases} 1, & \text{for } \beta = \mu \\ 0, & \text{otherwise} \end{cases} \quad (3.5)$$

is called the “Kronecker delta”. You also know it as the identity matrix

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.6)$$

Let us check Equation (3.4) explicitly with the matrix $a_{\alpha\beta}$ given in Equation (3.1):

$$\beta = 1, \mu = 1 \quad a_{11} a_{11} + a_{21} a_{21} = \cos^2 \theta + \sin^2 \theta = 1 \quad (3.7a)$$

$$\beta = 1, \mu = 2 \quad a_{11} a_{12} + a_{21} a_{22} = \cos \theta \sin \theta - \cos \theta \sin \theta = 0 \quad (3.7b)$$

A vector is defined to be a quantity that transforms just like the position vector under the same coordinate transformation, i.e., if any vector $\mathbf{A} = (A_1, A_2, A_3)$ in the coordinate system (x_1, x_2, x_3) transforms to $\mathbf{A}' = (A'_1, A'_2, A'_3)$ under the coordinate transformation given in Equation (3.2) then

$$A'_\alpha = a_{\alpha\beta} A_\beta \quad (3.8)$$

We can invert Equation (3.8) to obtain

$$A_\alpha = a_{\beta\alpha} A'_\beta. \quad (3.9)$$

In other words, the inverse of the matrix $a_{\alpha\beta}$ is its transpose. This is a property of the rotation matrix that you may already know. This is a consequence of demanding that the length of vectors remain unchanged under the transformation Equation (3.2).

Now let us try to find out how $s_{\alpha\beta}$ transforms under the same coordinate transformation. Start with Equation (2.11) which we now write in compact notation as

$$\mathbf{d}_\alpha = s_{\alpha\beta} \mathbf{r}_\beta \quad (3.10)$$

Here for notational simplicity we have used $\mathbf{r}$ instead of the symbol $\Delta \mathbf{r}$. As $\mathbf{d}$ is a vector, its transform $\mathbf{d}'$ will satisfy an equation like Equation (3.9). Hence

$$a_{\mu\alpha} \mathbf{d}'_\mu = s_{\alpha\beta} \mathbf{r}_\beta \quad (3.11)$$

Multiplying both sides by $a_{\kappa\alpha}$ and summing over the repeated index α and subsequent simplification we obtain

$$\mathbf{d}'_\kappa = a_{\kappa\alpha} a_{\gamma\beta} s_{\alpha\beta} \mathbf{r}'_\gamma \quad (3.12)$$

Multiply both sides of Equation (3.8) by $a_{\alpha\sigma}$ and sum over the repeated index α to get

$$\begin{aligned} a_{\alpha\sigma} A'_\alpha &= a_{\alpha\beta} a_{\alpha\sigma} A_\beta \\ \Rightarrow a_{\alpha\sigma} A'_\alpha &= \delta_{\beta\sigma} A_\beta = A_\sigma \\ \Rightarrow A_\sigma &= a_{\beta\alpha} A'_\beta. \end{aligned}$$

In the second line above we have used Equation (3.4).

In the transformed (primed) coordinate system we should have

$$\mathbf{d}'_{\kappa} = s'_{\kappa\gamma} \mathbf{r}'_{\gamma} \quad (3.13)$$

Comparing Equation (3.12) and Equation (3.13) we obtain the transformation law for the strain

$$\boxed{s'_{\kappa\gamma} = a_{\kappa\alpha} a_{\gamma\beta} s_{\alpha\beta}} \quad (3.14)$$

We have now proved that the strain is a second rank tensor. In general, any linear relationship between two vectors of the form

$$A_{\mu} = T_{\mu\nu} B_{\nu} \quad (3.15)$$

implies that the quantity $T_{\mu\nu}$ is a second rank tensor. This is sometimes called the “quotient rule”. This is a very useful way to find out what kind of tensor a physical quantity is. Two other examples where this rule is used are the moment of inertia tensor, or the polarizability tensor of a medium.

3.1.1 Fully anti-symmetric tensor

Let us revisit the displacement $\mathbf{d}_2$

$$\mathbf{d}_2 = \begin{pmatrix} 0 & \frac{a_{12}-a_{21}}{2} & \frac{a_{13}-a_{31}}{2} \\ \frac{a_{21}-a_{12}}{2} & 0 & \frac{a_{23}-a_{32}}{2} \\ \frac{a_{31}-a_{13}}{2} & \frac{a_{32}-a_{23}}{2} & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \quad (3.16)$$

where $a_{\alpha\beta} = \partial_{\beta} u_{\alpha}$. In Equation (2.9) we interpreted this equation as

$$\mathbf{d}_2 = (1/2)\Delta\boldsymbol{\phi} \times \Delta\mathbf{r} \quad (3.17)$$

To make a clearer connection to vorticity let us write

$$\mathbf{d}_2 = (1/2)\boldsymbol{\omega} \times \Delta\mathbf{r}\Delta t \quad (3.18)$$

where

$$\omega_1 = \partial_2 v_3 - \partial_3 v_2 \quad \omega_2 = \partial_3 v_1 - \partial_1 v_3 \quad \omega_3 = \partial_1 v_2 - \partial_2 v_1 \quad (3.19)$$

Clearly, Equation (3.16) can also be written as

$$(\mathbf{d}_2)_{\alpha} = (1/2)\Omega_{\alpha\beta} \Delta r_{\beta} \Delta t \quad (3.20)$$

where

$$\Omega_{\alpha\beta} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad (3.21)$$

By the quotient rule the set of number $\Omega_{\alpha\beta}$ constitutes a second rank tensor, that is anti-symmetric. The vector $\boldsymbol{\omega}$ and the second rank

The intermediate steps works out in the following manner:

$$\begin{aligned} d_{\alpha} &= s_{\alpha\beta} r_{\beta} \\ \Rightarrow a_{\mu\alpha} d'_{\mu} &= s_{\alpha\beta} r_{\beta} \\ \Rightarrow a_{\kappa\alpha} a_{\mu\alpha} d'_{\mu} &= a_{\kappa\alpha} s_{\alpha\beta} r_{\beta} \\ \Rightarrow \delta_{\kappa\mu} d'_{\mu} &= a_{\kappa\alpha} s_{\alpha\beta} r_{\beta} \\ \Rightarrow d'_{\kappa} &= a_{\kappa\alpha} s_{\alpha\beta} r_{\beta} \\ \Rightarrow d'_{\kappa} &= a_{\kappa\alpha} a_{\gamma\beta} s_{\alpha\beta} r'_{\gamma} \end{aligned}$$

I shall use either $\{x, y, z\}$ or $\{x_1, x_2, x_3\}$ or $\{x_{\mu}\}$ to denote Cartesian coordinate systems. I shall use bold fonts for vector, e.g., $\mathbf{A}$, $\Delta\mathbf{r}$ etc. I shall write the individual components of the vector $\mathbf{A}$ as either A_x, A_y, A_z , or A_1, A_2, A_3 or collectively A_{μ} . I shall typically use Greek indices to denote components of vectors or tensors. The convention can be extended in a straightforward way to denote any component of a higher rank tensor, e.g., $T_{\alpha\beta}$. Sometime I shall use the same symbol to denote the higher rank tensor itself. The symbols $\frac{\partial}{\partial x}$, ∂_x , ∂_1 will all mean the same thing, with the symbol ∂_{μ} denoting partial derivative with respect to a general coordinate x_{μ} .

tensor $\Omega_{\alpha\beta}$ can be related to each other by, you guessed it, a *third* rank tensor :

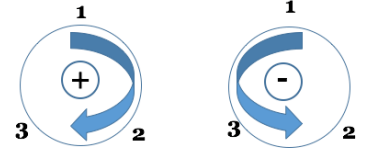
$$\Omega_{\alpha\beta} = \epsilon_{\alpha\beta\gamma} \omega_{\gamma} \quad (3.22)$$

This is called the Levi-Civita tensor

$$\epsilon_{\alpha\beta\gamma} = \begin{cases} 0, & \text{if any two of them are equal.} \\ 1, & \text{for even permutation} \\ -1, & \text{for odd permutation} \end{cases} \quad (3.23)$$

Using the Levi-Civita tensor the cross-product of two vectors can be written as

$$(\mathbf{A} \times \mathbf{B})_{\alpha} = \epsilon_{\alpha\beta\gamma} A_{\beta} B_{\gamma} \quad (3.24)$$



The only non-zero elements of $\epsilon_{\alpha\beta\gamma}$ are

$$\begin{aligned} \epsilon_{123} &= \epsilon_{231} = \epsilon_{312} = 1 \\ \epsilon_{132} &= \epsilon_{321} = \epsilon_{213} = -1 \end{aligned}$$

4 Stress

4.1 Stress

Let us now get back to the piece of “slime” we started the last lecture with. Consider a small volume element inside the slime. There are forces acting on this small volume element due to the rest of the slime. These forces act through the surface of the volume element. Let us be specific and consider a small box (a rectangular parallelepiped) of slime with its center at x, y, z , lying between $x - (1/2)\Delta x$ to $x + (1/2)\Delta x$, $y - (1/2)\Delta y$ to $y + (1/2)\Delta y$, and $z - (1/2)\Delta z$ to $z + (1/2)\Delta z$. Consider the face at $x + (1/2)\Delta x$. There is a force by the rest of the slime acting through this face. Let us call this force $\mathbf{f}(x + (1/2)\Delta x, y, z)$. In general, this force can point in any direction, i.e., it can have all the three components, $f_x(x + (1/2)\Delta x, y, z)$, $f_y(x + (1/2)\Delta x, y, z)$ and $f_z(x + (1/2)\Delta x, y, z)$. The force not only depends on space (i.e., x, y, z) but also which face we are considering. At the same point $x + (1/2)\Delta x, y, z$ we can consider two different planes, one perpendicular to the x axis, given by $\Delta y \times \Delta z$, and another perpendicular to the z axis, given by $\Delta y \times \Delta x$. The force $\mathbf{f}$ on these two infinitesimal area elements are clearly different. To emphasize that we label each component of the force accordingly. Each area element can be uniquely labeled by the direction of the unit vector normal to it. So the force components of the force acting at $(x + (1/2)\Delta x, y, z)$ on the face of the cube perpendicular to the x direction is labeled by σ_{xx} , σ_{yx} and σ_{zx} .

Next we define, the force per-unit-area as:

$$\sigma_{xx}(x + (1/2)\Delta x, y, z)\Delta y\Delta z = f_x(x + (1/2)\Delta x, y, z), \quad (4.1a)$$

$$\sigma_{yx}(x + (1/2)\Delta x, y, z)\Delta y\Delta z = f_y(x + (1/2)\Delta x, y, z), \text{ and } \quad (4.1b)$$

$$\sigma_{zx}(x + (1/2)\Delta x, y, z)\Delta y\Delta z = f_z(x + (1/2)\Delta x, y, z). \quad (4.1c)$$

Similarly we can define a set of nine number $\sigma_{\alpha\beta}$. More compactly, the force acting on an area element ΔS with its unit normal $\hat{\mathbf{n}}$ is given by

$$\mathbf{f}_\alpha = \sigma_{\alpha\beta} n_\beta \Delta S. \quad (4.2)$$

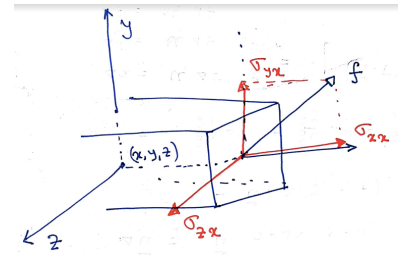


Figure 4.1: The force on the face $\Delta y\Delta z$ due to the rest of the material is $\mathbf{f}$. It has components along the three axis, σ_{xx} , σ_{yx} , σ_{zx} . If you consider a different face at the same point, e.g., $\Delta x\Delta y$, the force will be different.

As $\mathbf{f}$ is a vector, by the quotient rule $\sigma_{\alpha\beta}$ is a second rank tensor. This is called the “stress tensor”.

Let us try to calculate the total force on the volume element $\Delta x \Delta y \Delta z$. We have to sum over each component of the force through all the faces. Let us first sum the y component along the two faces perpendicular to the x axis:

$$\begin{aligned} f_y(x + \frac{1}{2}\Delta x, y, z) + f_y(x - \frac{1}{2}\Delta x, y, z) &= [\sigma_{yx}(x + (1/2)\Delta x, y, z) - \sigma_{yx}(x - (1/2)\Delta x, y, z)] \Delta y \Delta z \\ &= \left[\{ \sigma_{yx}(x, y, z) + \partial_x \sigma_{yx}|_{x,y,z} \frac{\Delta x}{2} + \dots \} - \{ \sigma_{yx}(x, y, z) - \partial_x \sigma_{yx}|_{x,y,z} \frac{\Delta x}{2} + \dots \} \right] \Delta y \Delta z \\ &= \partial_x \sigma_{yx} \Delta x \Delta y \Delta z + \dots \end{aligned}$$

Here we have expanded σ_{yx} in a Taylor series and truncated at the leading order term. Doing this for all the components over all the faces, we get the total force $\mathbf{F}$ on the volume element $\Delta V = \Delta x \Delta y \Delta z$ as

$$F_\beta = \partial_\alpha \sigma_{\beta\alpha} \Delta V \quad (4.4)$$

Here $\mathbf{F}$ is a vector and $\sigma_{\alpha\beta}$ is a second rank tensor, then by the quotient rule the components of the operator ∂_α forms a vector, this is the usual gradient operator

$$\nabla = (\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}) \quad (4.5)$$

Thus you may think of RHS of Equation (4.4) as the divergence of the stress tensor. You need to be careful about one point though, although the RHS of Equation (4.4) is often called a divergence it is not a scalar as usual divergence of a vector field is, but a vector; thus it is not invariant under rotation.

4.2 Equations of motion

By Newton's laws the force $\mathbf{F}$ is mass times acceleration of the small volume element ΔV . Let us now associate a density and a velocity to every point of a deformable body, $\rho(x, y, z)$ and $\mathbf{v}(x, y, z)$. The equation of motion then becomes

$$\rho \frac{dv_\alpha}{dt} = \partial_\beta \sigma_{\alpha\beta} \quad (4.6)$$

In a large class of elastic problems, we do not need to consider the dynamics but the statics – the body gets deformed but is not in motion. For such cases, Newton's law becomes the condition for mechanical equilibrium

$$\partial_\beta \sigma_{\alpha\beta} = 0 \quad (4.7)$$

everywhere in the body, which is merely a statement of force balance.

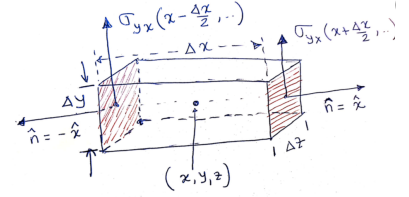


Figure 4.2: The force on the volume $\Delta x \Delta y \Delta z$ along the y direction due to the forces on the two faces that are at $x - \Delta x$ and $x + \Delta x$. The force on each face comes from the σ_{yx} component of the stress multiplied by the area-element $\hat{n} \Delta y \Delta z$. The unit vector $\hat{n} = \hat{x}$ for the face at $x + \Delta x$ and $\hat{n} = -\hat{x}$ for the face $x - \Delta x$.

4.3 Looking back

Let us now look critically at the results we have obtained.

There is an implicit assumption at the start of this chapter – the internal forces we consider are *local*. Clearly the internal forces appear due to forces between the molecules. We have assumed that such forces are *short-range*. This is, in most cases, a valid approximation as the molecules themselves are charge-neutral, hence the forces between them are dipole-dipole interaction which falls off very fast. As the theory of elasticity is a theory at scales much larger than molecular scales the forces between one part of a deformable body to another through the surface between them. This assumption may not hold when the deformation the body results in generating macroscopic electric field, e.g., for piezoelectric material. We will not consider such material.

Let us consider what happens at the boundary of a soft body. Start by ignoring any external volume force, e.g., gravity. Then the only way to deform a soft body is put some force on its surface. Assuming mechanical equilibrium, the force on the boundary is balanced by internal forces. Let us consider a small area element $\Delta y \Delta z$ at the boundary. Let the next external force be $\mathbf{P}$. Note that $\mathbf{P}$ need not be perpendicular to $\Delta y \Delta z$ it can point in any direction. This force must be balanced by the component of the internal stress, i.e., $P_\alpha + \sigma_{\alpha\kappa} \Delta y \Delta z = 0$. In general,

$$P_\alpha = \sigma_{\alpha\beta} n_\beta \Delta S. \quad (4.8)$$

If there are no external forces on the surface, then the internal stress must satisfy the boundary condition

$$\sigma_{\alpha\beta} n_\beta = 0 \quad (4.9)$$

at the surface of the soft body.

In addition to force-balance, mechanical equilibrium also implies torque-balance. Let us start by considering a small infinitesimal volume element (of volume ΔV). The torque on this volume element is:

$$\Gamma_\alpha \equiv \epsilon_{\alpha\beta\gamma} x_\beta F_\gamma = \epsilon_{\alpha\beta\gamma} x_\beta \partial_\mu \sigma_{\gamma\mu} \Delta V \quad (4.10)$$

Now we integrate over the full volume of the soft body to find out the total torque and write it as a surface integral and a volume integral:

$$\epsilon_{\alpha\beta\gamma} \int x_\beta \partial_\mu \sigma_{\gamma\mu} dV = \epsilon_{\alpha\beta\gamma} \oint_S x_\beta \sigma_{\gamma\kappa} n_\kappa dS - \int \epsilon_{\alpha\beta\gamma} \sigma_{\gamma\beta} dV = 0 \quad (4.12a)$$

Note that some authors [e.g., Landau and Lifshitz, 1970] define $\mathbf{P}$ as force per unit area thereby removing ΔS from the RHS of (4.8). We may do that somewhere later in these notes, but always after explicitly pointing this out.

Notice that the calculation that gives (4.4) has the same flavour as proof of Gauss's theorem. For any second rank tensor $A_{\alpha\beta}$, the Gauss's theorem can be generalised as

$$\oint_S A_{\alpha\beta} n_\beta dS = \int_V \partial_\beta A_{\alpha\beta} dV \quad (4.11)$$

where S is the closed surface that enclosed the volume V , and $\hat{n}$ is the unit normal to the surface element.

for the case where there are no external forces or torques on the soft body. Let us consider the surface integral first. If there is no external force on this soft body then at the boundary the surface integral must vanish due to (4.9). This implies that the volume integral, the second term in the middle of (4.12a), must also be zero. If it were not, the internal stresses would set up unbalanced torques on the body and angular momentum would not be conserved. Note that this implies that all three components of a vector are zero, i.e., $\alpha = 1, 2, 3$. This implies

$$\sigma_{\beta\gamma} = \sigma_{\gamma\beta}, \quad (4.13)$$

stress is a symmetric tensor. The symmetry is a consequence of angular momentum conservation.

There is another way to derive exactly the same result which is more pictorial as shown in Fig. 4.3. We look at the small square in x - y plane. Assume that the stresses vary very slowly and the cube is small enough such that any component of stress is constant on opposite side of the cube. The force balance in the x direction (or the y direction) is automatically satisfied.

$$\sigma_{xx} + \sigma_{xy} - \sigma_{xx} - \sigma_{xy} = 0 \quad (4.14)$$

Now sum over all the torques about z axis

$$a(2\sigma_{xy} - 2\sigma_{yx}) = 0 \quad (4.15)$$

where a is the length of a side of the square and the equality follows by torque balance. Again we prove the symmetry of the stress tensor.

Exercise 2

1. We set up the stress tensor from forces that acts on the faces of an infinitesimal cube. It turned out we needed 6 numbers (not nine because of symmetry of the stress tensor. Prove that these 6 (or 9) numbers are sufficient to give the internal force across any infinitesimal surface, not necessarily the face of a cube, inside a deformed body.
2. A soft body has an external force $\mathbf{P}$ acting on its surface. The body is in mechanical equilibrium. Show that the average stress, defined by

$$\langle \sigma_{\alpha\beta} \rangle \equiv \frac{1}{V} \int_V \sigma_{\alpha\beta} dV \quad (4.16)$$

is given by

$$\langle \sigma_{\alpha\beta} \rangle = \frac{1}{2V} \oint_S (P_\alpha x_\beta + P_\beta x_\alpha) dS \quad (4.17)$$

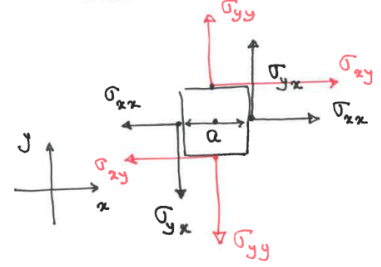


Figure 4.3: We look at the small square in x - y plane. Assume that the stresses vary very slowly and the cube is small enough such that any component of stress is constant on opposite side of the cube. In the figure stresses are denoted by $S_{\alpha\beta}$ where we have used the notation $\sigma_{\alpha\beta}$.

3. A soft body of uniform mass density ρ in a uniform gravitational field of acceleration $\gg$. Write the equations of mechanical equilibrium for its internal stresses.
4. **Is the stress tensor unique ?** Clearly the Newton's equations involves forces. Can there be two (or more) different stresses, i.e., two (or more) different second rank tensors that are functions of space, that gives the same internal forces (as a function of space) ? Let us define internal force per unit volume $\mathbf{f} = \mathbf{F}/\Delta V$ where $\mathbf{F}$ is given in (4.4). Then

$$f_\alpha = \partial_\beta \sigma_{\alpha\beta} \quad (4.18)$$

Now consider two different stresses, $\sigma_{\alpha\beta}$ and $\tilde{\sigma}_{\alpha\beta}$ such that

$$\sigma_{\alpha\beta} - \tilde{\sigma}_{\alpha\beta} = 2\partial_\gamma \psi_{\alpha\beta\gamma} \quad (4.19)$$

If both of these stresses give the same force, what kind of symmetry should $\psi_{\alpha\beta\gamma}$ have ? Do you see a similarity between this statement and gauge invariance in electrodynamics ?

5 Elastic Free Energy

5.1 Energy of deformation

Let us consider a deformed body with a deformation field $\mathbf{u}(\mathbf{x})$. The internal strain field is given by

$$s_{\alpha\beta} = \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha) \quad (5.1)$$

There is also an internal stress field $\sigma_{\alpha\beta}(\mathbf{x})$. This state of strain and stress may have resulted from external body forces, boundary forces and boundary conditions. For example, if the body is a bread dough then we can deform it by pressing it against the kitchen counter. The boundary forces are exerted by my hand and the kitchen counter.

This has developed an internal field of stress and strain. (For the special case of the bread dough, if I now remove my hand, the shape of the dough remain unchanged, it has developed some “permanent” deformation. But this point is moot for the discussion that follows.) Let us calculate the change in energy due to this small displacement. The elementary work done over a small volume element ΔV because of the virtual displacements is

$$\delta W(\Delta V) = \mathbf{F} \cdot \delta \mathbf{u} \quad (5.2a)$$

$$= F_\beta \delta u_\beta \quad (5.2b)$$

$$= (\partial_\alpha \sigma_{\alpha\beta} \Delta V) \delta u_\beta \quad (5.2c)$$

$$= (\partial_\alpha \sigma_{\beta\alpha} \Delta V) \delta u_\beta \quad (5.2d)$$

Integrating over the whole volume and simplifying¹ we obtain the total work done (for the infinitesimal volume ΔV) to be

$$\delta W = -\sigma_{\alpha\beta} \delta s_{\alpha\beta} \quad (5.3)$$

¹We need to remember two things: (a) δ and spatial derivative ∂_μ can be interchanged, i.e., they commute. and (b) The virtual displacements are zero on the boundaries, otherwise the boundary condition would have changed.

5.2 Thermodynamics

The elastic medium we are dealing with obviously consists of many many molecules and even the smallest scales we are looking at, for example the small volume δV are macroscopic scales, i.e., they

contain many many molecules. Thus the tool to describe them must be thermodynamics or equivalently statistical mechanics. Let me remind you that from the first and second law of thermodynamics we can write, quite generally,

$$TdS = dU + dW \quad (5.4)$$

where T is the temperature, S the entropy, U the internal energy and W the work done. For elastic bodies, dW is given by (5.3). For thermodynamic equilibrium this system must at the minima of its Helmholtz free energy $F = U - TS$. Rewriting the T - dS equation using the definition of F we obtain

$$dF = \sigma_{\alpha\beta} ds_{\alpha\beta} - SdT \quad (5.5)$$

Hence

$$\sigma_{\alpha\beta} = \left(\frac{\partial F}{\partial s_{\alpha\beta}} \right)_T \quad (5.6)$$

So far this equation is merely a consequence of thermodynamics and the definition of strain and stress. To be useful, we need to know how the free energy F depend on the strain tensor $s_{\alpha\beta}$.

5.3 Landau Free energy

Let me remind you how free energy is calculated in statistical mechanics. We first need to write down a Hamiltonian ($\mathcal{H}$) for the elastic body. Usually a Hamiltonian is a function of position and momentum of all the molecules in the small volume ΔV . To be able to write down the Hamiltonian we need to specify the potential between the atoms. This is a very difficult calculation and at the same time a model dependent one. So any conclusion we draw from such a calculation is also necessarily model dependent. Here we take an alternative approach.

We are trying to figure out how the free energy F depends on strain tensor $s_{\alpha\beta}$. As the zero of free energy is not physically important let us set $F = 0$ for $s_{\alpha\beta} = 0$. Then write the free energy as a Taylor expansion in $s_{\alpha\beta}$:

$$F = As_{\alpha\beta} + Bs_{\alpha\beta}s_{\mu\nu} + \text{h.o.t} \quad (5.7)$$

As F is a scalar, A must be a second rank tensor contracted with $s_{\alpha\beta}$. Next we remember that this an expansion about thermal equilibrium, at thermal equilibrium F must have a minima. Hence the derivatives $\partial F / \partial s_{\alpha\beta}$ must be zero. Hence A must be zero. Then up to leading order the free energy must have the form

$$F = E_{\alpha\beta\mu\nu} s_{\alpha\beta} s_{\mu\nu} \quad (5.8)$$

where $E_{\alpha\beta\mu\nu}$ is a fourth rank tensor that depends on the property of the material we are dealing with. We do not know what this tensor is. If we apply this to a homogeneous and isotropic material then $E_{\alpha\beta\mu\nu}$ must be an isotropic tensor.

A general isotropic fourth rank tensor can always be written as:

$$E_{\alpha\beta\mu\nu} = \lambda \delta_{\alpha\beta} \delta_{\mu\nu} + \mu (\delta_{\alpha\mu} \delta_{\beta\nu} + \delta_{\alpha\nu} \delta_{\beta\mu}) + \nu (\delta_{\alpha\mu} \delta_{\beta\nu} - \delta_{\alpha\nu} \delta_{\beta\mu}) \quad (5.9)$$

The total phenomenological free energy *density* of an 3D isotropic soft material is

$$F = F_0 + \frac{1}{2} \lambda s_{\alpha\alpha}^2 + \mu s_{\alpha\beta} s_{\alpha\beta} \quad (5.10)$$

where λ and μ are called the two Lamé coefficients. Separating strain into pure shear and pure volume change

$$s_{\alpha\beta} = (s_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} s_{\gamma\gamma}) + \frac{1}{3} \delta_{\alpha\beta} s_{\gamma\gamma} \quad (5.11)$$

we can rewrite the free energy density as

$$F = \mu (s_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} s_{\gamma\gamma})^2 + \frac{1}{2} K \frac{1}{3} \delta_{\alpha\beta} s_{\gamma\gamma}^2 \quad (5.12)$$

where μ is the shear modulus and $K = \lambda + \mu/3$ is the bulk modulus. Now let us calculate $\partial F / \partial s_{\alpha\beta}$.

To obtain the simplified form in the second line of (5.14b) notice that

$$\left(s_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} s_{\mu\mu} \right) \delta_{\alpha\beta} = 0 \quad (5.13)$$

$$dF = K s_{\mu\mu} ds_{\mu\mu} + 2\mu (s_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} s_{\mu\mu}) d(s_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} s_{\mu\mu}) \quad (5.14a)$$

$$= \left[K \delta_{\alpha\beta} s_{\mu\mu} + 2\mu \left(s_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} s_{\mu\mu} \right) \right] ds_{\alpha\beta} \quad (5.14b)$$

This implies

$$\sigma_{\alpha\beta} = \left(\frac{\partial F}{\partial s_{\alpha\beta}} \right)_T = K \delta_{\alpha\beta} s_{\mu\mu} + 2\mu (s_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} s_{\mu\mu}) \quad (5.15)$$

Equation 5.15 shows how stress depends on strain. As the dependence is linear this is equivalent to Hook's law.

Few comments are now in order.

1. Notice that we have not *derived* Hook's law. We essentially built it into our theory by assuming (5.10) where we assumed the free energy to depend on second power of strain. From thermodynamics it is clear that the leading order dependence must be quadratic thus it is clearly the Hook's law in inevitable in any elastic material for small enough deformation. But we have ignored term higher order in strain. Depending on what kind of material we are dealing with the Hookean regime may be either very small or reasonable large.

2. The fact that we have used only two elastic coefficients holds only for isotropic material. For other material the elastic tensor $E_{\alpha\beta\mu\nu}$ may need more parameters as is necessary by symmetry. For a simple cubic crystal it is also enough to have two elastic coefficients. See [Feynman et al., 1977, Chapter 39] for a proof.
3. To learn how to calculate the elements of the tensor of elasticity from a microscopic model of atoms for the simple case of a simple cubic crystal see [Feynman et al., 1977, Chapter 39].
4. The style of writing down the free energy based on the symmetry of the problem (in this case isotropy and homogeneity) and from basic thermodynamic argument was popularised by Lev Landau. In his honour similar expression for free energy are often called Landau free energy.

5.3.1 Elastic coefficients

The elements of the tensor $E_{\alpha\beta\mu\nu}$ are called elastic coefficients or elastic moduli. As we are dealing with isotropic material we need only two λ and μ or equivalently K and μ .

In a state of thermodynamic equilibrium the free energy is a minimum. If no external forces act on the body then F as function of $s_{\alpha\beta}$ must have a minimum for $s_{\alpha\beta} = 0$. Hence the quadratic form in (5.10) and (5.12) must be positive. In (5.12) consider the case where $s_{\kappa\kappa} = 0$. The only the first term remains which is a complete square. Then $\mu > 0$. Similarly consider a case where the strain has no shear deformation but only volume deformation. Then by a similar argument $K > 0$.

5.3.2 From strain to stress

As (5.15) is linear it is essentially a matrix equation which can be inverted to write the strain as a function of stress.

$$s_{\alpha\beta} = \delta_{\alpha\beta} \frac{\sigma_{\kappa\kappa}}{9K} + \frac{1}{2\mu} \left(\sigma_{\alpha\beta} - \frac{1}{3} \delta_{\alpha\beta} \sigma_{\kappa\kappa} \right) \quad (5.17)$$

Multiply (5.15) by $\delta_{\alpha\beta}$ on both sides to obtain (remember the repeated indices are summed, and $\delta_{\alpha\beta} \delta_{\alpha\beta} = 3$).

$$\sigma_{\alpha\alpha} = 3Ks_{\mu\mu} + 2\mu \left(s_{\mu\mu} - \frac{1}{3} 3s_{\mu\mu} \right) \quad (5.16a)$$

$$= 3Ks_{\mu\mu} \quad (5.16b)$$

Substitute this back in (5.15).

6 Elastostatics : the Navier-Cauchy Equations

6.1 Derivation

We now have all the ingredients to write down the equations that describes the elastic deformations. The first piece is a consequence of the Newton's law for a small volume ΔV in a soft body under static equilibrium, the sum of all internal and external forces are zero:

$$\partial_\beta \sigma_{\alpha\beta} + f_\alpha^{\text{ext}} = 0. \quad (6.1)$$

Here the divergence of the stress includes all the internal forces and f^{ext} is the sum of all external *force densities*. Note that f^{ext} is not a force but a force density, force-per-unit-volume. Substituting stress-strain relation

$$\sigma_{\alpha\beta} = E_{\alpha\beta\mu\nu} s_{\mu\nu} \quad (6.2)$$

in (6.1) we obtain a partial differential equation (PDE) for the displacement field $\mathbf{u}$.

$$E_{\alpha\beta\mu\nu} \partial_\beta s_{\mu\nu} + f_\alpha^{\text{ext}} = 0. \quad (6.3)$$

This is the PDE which together with the boundary condition determines the displacement field in a soft body. If we now use the expression for the tensor of elasticity for isotropic material then we obtain

$$(\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \mu \nabla^2 \mathbf{u} = -\mathbf{f}_\alpha^{\text{ext}} \quad (6.4)$$

This is the Navier-Cauchy equation.

6.2 What is the nature of the beast ?

The Navier-Cauchy equation is a second order vector partial differential equation. The version with the force on the RHS is the inhomogeneous version and one without is the homogeneous one.

This is a linear second order inhomogeneous equation. As the PDE is linear, following properties follow in a straightforward manner:

1. **Uniqueness of solutions:** Given a set of boundary conditions the solutions are unique. That means, if you can somehow guess a solution and show that it actually satisfies the equation and its boundary condition, voila, you do not need to look any further, you have found *the* solution.
2. **Superposition of solutions:** If there are two solutions then a linear combination of the solutions is a solution. You may remember that this is used very imaginatively to find solutions of Laplace/Poisson equations – a trick called method of images – in electrostatics.
3. **Symmetry:** If the problem with boundary conditions have certain symmetry then the solutions will have the same symmetry.

The last of these requires some exposition.

Let me start by reminding you of a similar problem you have solved in your electrodynamics course. There is a metal sphere of radius R at the center of our coordinate system. Far away from the sphere there is a constant electrostatic field $\mathbf{E}$ along the z direction. The task is to solve for the electrostatic field (or equivalently the electrostatic potential, ϕ) everywhere. To solve for the potential we need to solve for the Laplace's equation

$$\nabla^2 \phi = 0 \quad (6.5)$$

with the appropriate boundary condition. As the problem is axially symmetric we start stating that ϕ is a function of radial coordinate r and angle θ but not the azimuthal angle ϕ . This is a typical example of how the symmetry of the boundary conditions determine the symmetry of the solution.

In contrast now consider flow of water around a sphere. If the flow speed is small the flow is indeed axially symmetric. But as the flow speed increases the flow loses its symmetry, even becomes time dependent and eventually turbulent. Why does the symmetry argument fail in this case? Because the partial equation that determines the flow is a nonlinear equation, the Navier–Stokes equation. For a nonlinear equation there may exist solutions which do not obey the symmetry of the boundary conditions, indeed there are typically many many solutions.

6.2.1 Boundary Condition

To derive the boundary condition let us consider a small pillbox centered at the surface of a soft body, see Figure. 6.2 We will eventually take the limit of the height of this pillbox to zero. The surface of the

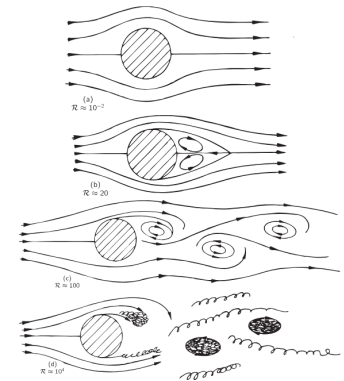


Figure 6.1: A sketch of typical flow around a sphere. The flow speed increases from top to bottom, everything else remains constant. You can see how the symmetries are low. Figure are from [Feynman et al., 1977, Chapter 41]

pillbox is so small that the stress tensor remains a constant over the surface. Let us integrate (6.1) over this pillbox:

$$\int dV \partial_\beta \sigma_{\alpha\beta} + \int dV f_\alpha^{\text{ext}} = 0 \quad (6.6)$$

By the Gauss's theorem the first interal can be converted to a surface integral over the surface of the pillbox. The sides do not contribute because we are going to take the limit that the height of the pillbox goes to zero. The only contribution comes from the top and the bottom surface. The top surface is in the region marked by "out" and the bottom surface is in the region marked by "in". Thus we obtain

$$\delta S (\sigma_{\alpha\beta}^{\text{o}} - \sigma_{\alpha\beta}^{\text{i}}) n_\beta + \int dV f_\alpha^{\text{ext}} = 0, \quad (6.7)$$

where δS is the area of the top and bottom face of the pillbox.

If the external force is a volume force, then as the height of the pillbox goes to zero the volume integral also goes to zero. But if the external force is a surface force then it is singular over the volume, concentrated at the surface. To be specific let us take the orgin of our coordinate system at the surface and take the z axis perpendicular to the surface at the center of the pillbox. Then the external force can be written as

$$\mathbf{f}^{\text{ext}} = \mathbf{P} \delta(z) \quad (6.8)$$

where $\delta(z)$ is the Dirac delta function. Substituting this in (6.7) we obtain:

$$(\sigma_{\alpha\beta}^{\text{o}} - \sigma_{\alpha\beta}^{\text{i}}) n_\beta + P_\alpha = 0, \quad (6.9)$$

Here we have used the symbol $\mathbf{P}$ to emphasize that the force we consider is force-per-unit-area or has the same dimension as pressure although it does not necessarily act perpendicular to the surface. In some case, the external stress is so small compared to the internal stress, we can set $\sigma_{\alpha\beta}^{\text{o}} = 0$ and obtain

$$\sigma_{\alpha\beta}^{\text{i}} n_\beta = P_\alpha \quad (6.10)$$

If there are no surface forces on the interface, the normal component of the stress is continuous across the boundary.

The Navier-Cauchy equation is for the displacement vector $\mathbf{u}$, thus this kind of boundary conditions essentially set a condition on the derivatives of $\mathbf{u}$ at the boundary and the condition actually involves elastic modulii of the material itself. Equation 6.9 is actually three equation, one for each value of α packed into one. This is the **Neumann boundary condition**, similar to specifying the electric field at the boundary in electrostatics problems.

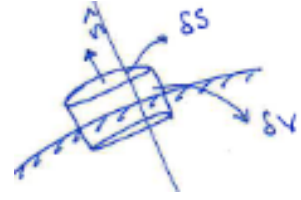


Figure 6.2: A small pillbox around the boundary.

A different boundary condition can be set on the deformation field itself. For example, if a soft body is confined by a rigid boundary then an appropriate boundary condition is $\mathbf{u} = 0$ – this is the **Dirichlet boundary condition**.

6.3 Uniform uniaxial strain

Let us now apply what we have learnt to the simplest possible case. A rod with a rectangular crosssection is acted upon on both sides by the same force $F\hat{z}$, see Figure. 6.3. The cross-section of the rod is constant hence the force per-unit-area is the same on every crosssection. As the stress is uniform, the strain must also be uniform.

Let us now apply the boundary conditions to the side boundaries, particularly to the surface whose normal is the unit vector along the x axis $\hat{x}$. In (6.9) $\hat{n} = (1, 0, 0)$, $P = 0$, and the stress outside is also zero as we ignore the small stress that comes from air pressure. Hence we obtain

$$\sigma_{11} = 0 \quad \text{and} \quad \sigma_{12} = 0. \quad (6.11)$$

Applying the same arguments to the surface element whose normal is the $\hat{y}$ vector we obtain

$$\sigma_{21} = 0 \quad \text{and} \quad \sigma_{22} = 0. \quad (6.12)$$

We also assume that the cross-section of the rod is so small that the stress does not vary over it. Hence all through the cross-section (6.11) and (6.12) holds. Now we apply the same boundary condition to the surface at one of the ends. Here the normal vector to the surface is $\hat{n} = \hat{z} = (0, 0, 1)$. The boundary condition then takes the form

$$\sigma_{33} = P \quad \text{and} \quad \sigma_{23} = 0 \quad \text{and} \quad \sigma_{13} = 0 \quad (6.13)$$

From (6.11), (6.12) and (6.13) we obtain $\sigma_{\mu\mu} = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{33} = P$. Expressing the strain as a function of the stress, (5.17) we obtain

$$s_{33} = \frac{1}{9K}\sigma_{33} + \frac{1}{2\mu}\left(\sigma_{33} - \frac{1}{3}\sigma_{33}\right) = \left(\frac{1}{9K} + \frac{1}{3\mu}\right)P \quad (6.14a)$$

$$s_{11} = \frac{1}{9K}\sigma_{33} + \frac{1}{2\mu}\left(\sigma_{11} - \frac{1}{3}\sigma_{33}\right) = \left(\frac{1}{9K} - \frac{1}{6\mu}\right)P \quad (6.14b)$$

$$s_{22} = s_{11} \quad (6.14c)$$

Let me now remind you two emperical results that you may remember from your high school. For a rod stress in the way I have just described stress divided by strain is a property of the material of the rod, called Young's modulus, Y . And for a rod strained along the z direction is also strained along the two orthogonal directions, x and y .



Figure 6.3: Sketch of a rod with two equal and opposed stresses applied on the two ends. The length of the rod is L . We choose the z axis to be along the axis of the rod.

Typically if the length of the rod increases along z its width decreases along x (or y). The ratio of these two strains is typically negative, and denoted by the Poisson's ratio, σ . Hence from (6.14) we obtain

$$Y = \frac{P}{s_{33}} = \frac{9K\mu}{3K + \mu} \quad (6.15a)$$

$$\sigma = -\frac{\sigma_{11}}{\sigma_{33}} = \frac{1}{2} \frac{3K - 2\mu}{3K + 2\mu} \quad (6.15b)$$

6.3.1 Elastic moduli

Thus we have arrived at three different pairs of elastic moduli. One, the two Lamé coefficients, λ and μ ; Two, the bulk modulus K and the shear modulus μ ; and three, the Young's modulus Y and the Poisson's ratio σ . The three pairs are related to each other. All of the the elastic moduli are positive except λ and σ . The fact that K and μ are positive follows from thermodynamic stability. From that it follows Y must also be positive. But there is no additional constraint on either λ or σ . In principle, the Poisson's ratio σ lies between -1 for ($K = 0$) and $1/2$, for $\mu = 0$ or $K \rightarrow \infty$, which corresponds to effectively incompressible material. In practice I know of no material for which σ is negative. If you assume $\sigma > 0$, it also follows that $\lambda > 0$. Thus in practice both σ and λ are positive.

Finally, we collect here the Free energy, Hook's Law and Navier–Cauchy equations written with the elastic coefficients Y and σ

$$F = \frac{Y}{2(1 + \sigma)} \left(s_{\alpha\beta}^2 + \frac{\sigma}{1 - 2\sigma} s_{\kappa\kappa}^2 \right) \quad (6.16a)$$

$$\sigma_{\alpha\beta} = \frac{Y}{1 + \sigma} \left(s_{\alpha\beta} + \frac{\sigma}{1 - 2\sigma} s_{\kappa\kappa} \delta_{\alpha\beta} \right) \quad (6.16b)$$

$$Y s_{\alpha\beta} = (1 + \sigma) \sigma_{\alpha\beta} - \sigma \sigma_{\kappa\kappa} \delta_{\alpha\beta} \quad (6.16c)$$

$$\frac{1}{1 - 2\sigma} \nabla(\nabla \cdot \mathbf{u}) + \nabla^2 \mathbf{u} = -\frac{2(1 + \sigma)}{Y} \mathbf{f} \quad (6.16d)$$

where $\mathbf{f}$ is the external force *per unit volume*.

6.4 Equations of elasticity

So far in this chapter we have considered our soft body to be in *static* mechanical equilibrium. It is straightforward to apply the same equations to the case where there the soft body is moving. Simple application of Newton's laws to a small volume element ΔV , with

density $\rho(\mathbf{x}, t)$ and velocity $\mathbf{v}(\mathbf{x}, t)$, gives:

$$\rho \frac{dv_\alpha}{dt} = \partial_\beta \sigma_{\alpha\beta} + f_\alpha^{\text{ext}} \quad (6.17a)$$

$$= E_{\alpha\beta\mu\nu} \partial_\beta s_{\mu\nu} + f_\alpha^{\text{ext}} \quad (6.17b)$$

$$\Rightarrow \rho \frac{d\mathbf{v}}{dt} = (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \mathbf{f}^{\text{ext}}. \quad (6.17c)$$

The last expression is valid for isotropic elastic material.

7 Solutions of the Navier-Cauchy Equations

Now that we have derived the equations of elasticity let us study its consequence by solving them.

As the equations are linear we can start by finding out the Green's function.

7.1 Green's function

7.1.1 Background

Let me remind you what Green's function is. Consider a linear inhomogeneous differential equation

$$\mathcal{L}u = f, \quad (7.1)$$

where $\mathcal{L}$ is a differential operator, $u(x)$ is a field in one dimension and $f(x)$ is typically called the source. We seek a solution of this equation with the form

$$u(x) = \int G(x, y) f(y) dy. \quad (7.2)$$

This equation essentially defined the Green's function $G(x, y)$. Substitute (7.2) in (7.1) to obtain

$$\mathcal{L}u = \int dy \mathcal{L}(x) G(x, y) f(y) = f(x), \quad (7.3)$$

where the last equality follows from (7.1). Let me remind you the definition of Dirac's delta function, $\delta(\cdot)$,

$$\int \delta(x - y) f(y) dy = f(x) \quad (7.4)$$

Comparing (7.4) with (7.3) we obtain

$$\mathcal{L}(x) G(x, y) = \delta(x - y) \quad (7.5)$$

The Green's function is a solution of the equation $\mathcal{L}u = 0$ with a delta-function source on the right hand side. The advantage of this approach is, once we know the Green's function we know can write down the solution of the differential equation (7.1) for any source in the form of an integral, as in (7.2) – we have solved the differential equation for any source. Obviously, (7.5) must be solved for the same boundary conditions that hold for (7.1).

Note that, depending on G and f , often the integral in (7.2) cannot be written down in a close form. However that does not change the conclusion that once we have the Green's function we have solved the problem.

Let me give you an example that you know very well. Consider the Poisson's equation of electrostatics in vacuum.

$$\nabla^2 \phi = \frac{1}{\epsilon_0} \rho, \quad (7.6)$$

where $\phi(\mathbf{x})$ is the electrostatic potential, $\rho(\mathbf{x})$ is the charge density and ϵ_0 is the permittivity of vacuum. The solution to this equation is already known to you from Coulomb's law:

$$\phi(\mathbf{x}) = \frac{1}{\epsilon_0} \frac{1}{4\pi} \int d\mathbf{y} \frac{\rho(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \quad (7.7)$$

To write down the Green's function we change the source, the charge density, to a delta function, $\delta(\mathbf{y})$ to obtain

$$G_{\text{electrostatics}}(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{y}|} \quad (7.8)$$

7.1.2 Definition of problem

To obtain the Green's function of the Navier–Cauchy equation we consider the specific situation where we have a source $\mathbf{f}$ at the origin and we choose the z direction of our coordinate system along $\mathbf{f}$. We write down the equations with elastic moduli, Υ and σ :

$$\frac{1}{1-2\sigma} \nabla(\nabla \cdot \mathbf{u}) + \nabla^2 \mathbf{u} = -\frac{2(1-\sigma)}{\Upsilon} \mathbf{f} \delta(\mathbf{x}) \quad (7.9)$$

The solution must have boundary conditions at infinity. We assume that the deformation goes to zero at infinity. By symmetry, here, the solution of the Navier–Cauchy equation can depend only on the radial coordinate r and the angular coordinate θ but not on the azimuthal coordinate ϕ .

For any general force (density) $\mathbf{f}(\mathbf{x})$ on the right hand side we expect the solution can be written in the form

$$u_\alpha(\mathbf{x}) = \int G_{\alpha\beta}(\mathbf{x}, \mathbf{y}) f_\beta(\mathbf{y}) d\mathbf{y}. \quad (7.10)$$

The Green's function is a second rank tensor.

The derivation of the Green's function or the solution of (7.9) is typical of many such mathematical derivation. It requires certain tricks that you need to remember. The first trick is to write

$$\mathbf{u} = \mathbf{v} + \mathbf{w} \quad (7.11a)$$

$$\text{such that } \nabla^2 \mathbf{v} = -\frac{2(1-\sigma)}{Y} \mathbf{f} \delta(\mathbf{x}) \quad (7.11b)$$

The rationale is that we divide $\mathbf{u}$ into two parts such that we can package one part as the Poisson equation whose solutions is already known to us. Remember that as we are dealing with linear equations superposition of solutions are valid thereby I can decompose $\mathbf{u}$ into any two pieces that I like. Also as uniqueness of solutions hold, once I have found a solution, by any method whatsoever including wild guesses, it is the solution.

The second line of (7.11b) are three Poisson equations one for each component of $\mathbf{v}$. The three solutions written in compact vector form are:

$$\mathbf{v} = -\frac{2(1-\sigma)}{4\pi Y} \left(\frac{1}{r} \right) \mathbf{f} \quad (7.12)$$

Substituting (7.11b) in (7.9) we obtain

$$\frac{1}{1-2\sigma} \nabla \nabla \cdot (\mathbf{v} + \mathbf{w}) + \nabla^2 \mathbf{w} = 0 \quad (7.13)$$

Taking its curl, we obtain $\nabla^2 (\nabla \times \mathbf{w}) = 0$. This can be satisfied by assuming $\nabla \times \mathbf{w} = 0$, which implies that $\mathbf{w}$ can be written as the gradient of a scalar potential

$$\mathbf{w} = \nabla \Phi \quad (7.14)$$

Substituting this back into (7.13) we obtain

$$\frac{1}{1-2\sigma} \nabla^2 (\nabla \Phi) = -\nabla (\nabla \cdot \mathbf{v}). \quad (7.15)$$

As we know $\mathbf{v}$, (7.12), we know the RHS of (7.15). Substituting (7.12) in (7.15) and simplifying we obtain

$$\nabla^2 \Phi = \frac{1+\sigma}{4\pi Y(1-\sigma)} (\mathbf{f} \cdot \nabla) \frac{1}{r} \quad (7.16)$$

The next step is the second trick. Consider the equation

$$\nabla^2 \Psi = \frac{1}{r} \quad (7.17)$$

For any constant $\mathbf{f}$ it follows that

$$\nabla^2 (\mathbf{f} \cdot \nabla) \Psi = \mathbf{f} \cdot \nabla \left(\frac{1}{r} \right) \quad (7.18)$$

Thereby we can rewrite (7.16) as

$$\nabla^2 \Phi = \frac{1 + \sigma}{4\pi Y(1 - \sigma)} \nabla^2 (\mathbf{f} \cdot \nabla) \Psi. \quad (7.19)$$

A solution is

$$\Phi = \frac{1 + \sigma}{4\pi Y(1 - \sigma)} \mathbf{f} \cdot \nabla \Psi \quad (7.20)$$

All we need to do now is to solve (7.17). Clearly Ψ is a function of r alone. Using the expression for the Laplacian operator in spherical polar coordinate system we obtain the differential equation

$$\frac{1}{r} \frac{d^2}{dr^2} (r\Psi) = \frac{1}{r}, \quad (7.21)$$

whose solutions is

$$\Psi = A + \frac{B}{r} + \frac{r}{2} \quad (7.22)$$

The second term is singular at $r = 0$. Thus we set $B = 0$. Note that, although there is a singular part in the RHS of the Navier–Cauchy equation we are dealing with, the singular part is attributed to $\mathbf{v}$ not $\mathbf{w}$. Thus $\mathbf{w}$ must remain well behaved everywhere in the domain including $r = 0$. Furthermore, we do not need to fix A because it is not Ψ but its derivatives that appear in (7.20). Substituting (7.22) to (7.20) we obtain

$$\Phi = \frac{1 + \sigma}{8\pi Y(1 - \sigma)} (\mathbf{f} \cdot \nabla) r \quad (7.23)$$

We now have all the pieces, (7.11b), (7.12), (7.14) and (7.23), to write down the solution of (7.9). On writing the result in the form (7.10) we obtain the Green's function as

$$G_{\alpha\beta}(\mathbf{r}) = \frac{1}{4\pi\mu} \left[\frac{\delta_{\alpha\beta}}{r} - \frac{1}{4(1 - \sigma)} \frac{\partial^2 r}{\partial x_\alpha \partial x_\beta} \right] \quad (7.24)$$

In principle, we have now solved the Navier–Cauchy equation for all problems. It is important to notice that the Green's function falls off as one over the distance like the Coulomb potential of a point charge and consequently, strain, which is the derivative of the displacement falls off as $\sim 1/r^2$, like the electrostatic field of a point charge.

7.1.3 A single concentrated surface force on infinite half-plane

This can also be solved by a generalization of the same trick in the previous subsection. See e.g., Ref. [Landau and Lifshitz, 1970, section 8]

7.2 Spherical and cylindrical problem

Many such problems in elasticity can be simplified by using symmetry. As concrete example let us first consider the problem of a spherical shell.

7.2.1 A spherical shell

We consider a spherical shell made of linear elastic material with inner radius R_1 and outer radius R_2 . The inner pressure is p_1 and outer pressure is p_2 . In this case writing the Navier–Cauchy equation in a slightly different form helps. First notice that there is no external volume forces. Hence we can use (6.16d) with the right hand side set to zero. Then we rewrite the rest of (6.16d) in the following form

$$\nabla \nabla \cdot \mathbf{u} - \frac{1-2\sigma}{2(1-\sigma)} \nabla \times \nabla \times \mathbf{u} = 0 \quad (7.26)$$

To arrive at this form use the vector identity that for a vector field $\mathbf{A}$,

$$\nabla \times \nabla \times \mathbf{A} = \nabla \nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A} \quad (7.25)$$

The symmetry of the problem dictates that the displacement field is radial everywhere, hence its curl must be zero. Thereby we arrive at the partial differential equation

$$\nabla \nabla \cdot \mathbf{u} = 0 \quad (7.27a)$$

$$\Rightarrow \nabla \cdot \mathbf{u} = \text{constant} \quad (7.27b)$$

You can now use the expression for the divergence operator in spherical polar coordinate system with the additional simplification that $\mathbf{u}$ is a function of only the radial coordinate: $\mathbf{u} = (\hat{r}u(r), 0, 0)$ and obtaining

$$u(r) = Ar + \frac{B}{r^2} \quad (7.28)$$

The two constant A and B must be fit by the boundary conditions. To do that calculate the strain tensor. From the strain tensor, using Hook's law, (6.16b), calculate the stress tensor. The boundary condition at $r = R_1$ and $r = R_2$ are respectively,

$$\sigma_{rr} = -p_1 \quad \text{and} \quad \sigma_{rr} = -p_2 \quad (7.29)$$

By matching the boundary conditions you can obtain A and B . This is a typical example of a series of steps you take to solve the Navier–Cauchy equation in cases where there is symmetry.

1. Identify the symmetry and write the Navier–Cauchy equation in a form that is most suited to your problem. Example: (7.26).
2. Reduce the PDE to one or more ODEs. Example (7.28). And solve this ODE. You will typically have two constants of integration.
3. From the solution calculate the strain. From strain calculate the stress. Using the (6.9) match the boundary conditions and calculate the constant. In some cases you may have a boundary condition directly on $\mathbf{u}$ itself, e.g., cases where elastic material is bounded by rigid boundary.

4. In some cases you may need to use different boundary conditions, e.g., the displacement must be zero and infinity (as I have done in deriving the Green's function), or the displacement must remain finite everywhere in the domain including the origin.

There are several examples below where these steps are applied.

7.2.2 *A cylindrical shell*

7.2.3 *A cylinder rotating about its own axis*

7.2.4 *A pressurized spherical hole in an infinite medium*

7.2.5 *A deformed medium with a spherical hole inside*

7.2.6 *How valid are the solutions ?*

Now that you have done several mathematical exercises with the Navier–Cauchy equation let us look at whether the solutions we have just calculated holds under all conditions. A particularly interesting case of the solutions for the spherical shell is a thin shell, e.g., a football. According to our solution the problem always remain spherically symmetric. Thus as we deflate a football – decrease p_1 the football should deflate but always remain spherically symmetric. But from your experience you know that the football does not remain spherically symmetric, it buckles ! How did we get this wrong ?

7.3 *Relation to the bi-harmonic equation*

There is a connection between the Navier–Cauchy equation and the bi-harmonic equation that is useful to know, particularly to solve problems in elasticity for cases where the displacement is a two-dimensional.

8 Contact Mechanics

8.1 Green's function

The Green's function is a second rank tensor. For a external δ function force *per unit volume* at the origin pointing along the direction $\mathbf{n}$, the Green's function is

$$G_{\alpha\beta}(\mathbf{r}) = \frac{1}{4\pi\mu} \left[\frac{\delta_{\alpha\beta}}{r} - \frac{1}{4(1-\sigma)} \frac{\partial^2 r}{\partial x_\alpha \partial x_\beta} \right] \quad (8.1)$$

The trick necessary to obtain this solution is to decompose the solution of the Navier–Cauchy equation to two vector field

$$\mathbf{u} = \mathbf{v} + \mathbf{w} \quad (8.2a)$$

$$\text{such that } \nabla^2 \mathbf{v} = -\frac{2(1+\sigma)}{Y} \mathbf{f} \delta(\mathbf{r}) \quad (8.2b)$$

$$\text{and } \mathbf{w} = \nabla \phi \quad (8.2c)$$

where ϕ is a scalar function of position and $\mathbf{f}$ is a constant vector. This decomposition can always be done. This decomposition is often an useful trick to solve the Navier–Cauchy equation.

8.2 Solid bodies in contact

A very practical application of three dimensional elasticity we have learnt so far is solid bodies in contact, for example a solid ball hitting a surface. Such problems were first solved by Heinrich Hertz over a Christmas break when he was twenty three years old. Contact mechanics is a vast field, I will touch on just the basic idea.

8.2.1 Two Hertzian spheres

Two spheres of Young's modulus Y_1 and Y_2 and radii R_1 and R_2 are pushed towards each other by an amount h with a force F along the axis the connects the centers of the spheres. How does F depends on h ? You can imagine how, in principle, the solution of this problem would proceed. There would be an area of contact between the

sphere. This is one of the unknowns of the problem. Presumably the area would change with F . We must solve the Navier-Cauchy equation for each of the spheres individually. Then the solutions must be matched across the area of contact such that the normal stress is balanced – otherwise the system is not in mechanical equilibrium. But, in principle, to solve the Navier-Cauchy I also need to know the boundary conditions all over both the spheres, exactly how are they being pushed against each other. For example, for two colliding balls, the boundary conditions away from the contact is one where the stress must go to zero.

Based on geometry and dimensional estimates we can show that the force–distance relationship for small deformations is

$$h \propto \frac{F^{2/3}}{Y_h^{2/3} R_h^{2/3}} \quad (8.3a)$$

$$\text{where } \frac{1}{Y_h} \equiv \frac{1}{Y_1} + \frac{1}{Y_2} \quad (8.3b)$$

$$\text{and } \frac{1}{R_h} \equiv \frac{1}{R_1} + \frac{1}{R_2} \quad (8.3c)$$

$$\text{with } a \propto \frac{R_h^{1/3} F^{1/3}}{Y_h^{1/3}} \quad (8.3d)$$

where a is the radius of their circle of contact. This result is, strictly speaking, not correct. A more careful calculation shows that the Young's moduli must be replaced by $Y/(1 - \sigma^2)$.

8.2.2 Generalization

The result can be generalized to any two surfaces in contact. The exponent $h \propto F^{2/3}$ and $a \propto F^{1/3}$ remains the same. The rest is replaced by a complex expression involving the four radii of curvature.

These results form the basis of measurement of elastic moduli of soft material, e.g., brain tissue, cell, by atomic force microscope (AFM) experiments.

8.2.3 Bulging and anti-Hertz

Consider an elastic material that can bulge through a hole at the origin but otherwise confined to the negative half-plane by a rigid surface, see Figure 8.2.3. This is called the bulging problem.

A similar problem appears if instead of the Hertz problem you push down on a soft material with a washer – a ring with a hole in it. The soft material will rise like a dimple through the hole. This is the anti-Hertz problem.

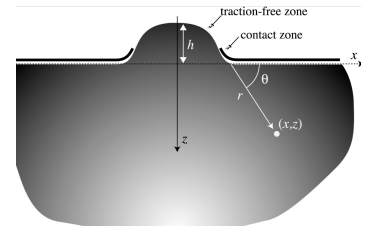


Figure 8.1: The bulging problem

8.3 *Brain as a soft material*

9 Bent plates

We now want to write down the theory of elasticity for two-dimensional material, for example, a piece of paper. Importantly, this is a lower dimensional material embedded in a higher dimension. The displacement vector field, which is the fundamental quantity of elasticity, is now a function of this two dimensional manifold. But the displacement vector itself have not two but three components. A general displacement of a point on a piece of paper can be decomposed into two components, one displacement perpendicular to the paper itself and the other, which in itself is a two-dimensional vector, along the paper. Henceforth we shall call this components (u_1, u_2, w) respectively, where the two dimensional vector (u_1, u_2) lies on the two dimensional manifold and w is the displacement perpendicular to it.

9.1 From 3D to 2D

We start with the three dimensional elasticity we already know and apply it to a domain in which one dimension is very thin compared to other two. This will allow us integrate over the thin dimension to write down a theory of effective elasticity in two-dimensions. This is a particular example of many similar approximations often used in soft-matter physics, hence pay close attention.

We start by considering the simplest case where the a thin plate is bent by a small amount. What we want to do, is to start with the three dimensional elastic free energy that we had already written down and integrate it over the thickness of the plate such that we can write the integrated free energy as a function of the displacement vector u_1, u_2, w .

9.1.1 Neutral surface

As you bend a thin sheet you can imagine that the plane on one side gets contracted and the opposite side gets extended. Then there must be a plane that does not change in length at all. For large bending we expect there must always be a surface that is not stretched at all



Marie-Sophie Germain, 1776 to 1831, was a French mathematician who made pioneering contributions to theory of elasticity. She came from a wealthy family (her father was formally a bourgeoisie). She was a teenager at the time of the French revolution had very little formal education in mathematics as was usual for women at those times in Europe. She learnt from correspondence with Lagrange, Legendre and Gauss. Her work on elasticity was one of the first attempts to write down the equations for vibrations of a plate. She was the first woman to win the grand prize of the Paris academy of sciences for this work. Contemporary mathematicians considered her work to suffer from an absence of rigor, so possibly she was more a physicist than a mathematician in this field. She also made important contributions to number theory.

but this surface may have quite a complex geometry. But for small bending the cross-section of this surface is a parabola. This is called the *neutral plane* because it is neither stretched or squeezed. Note that this is a physical object, not merely a mathematical construction. All the planes on one side of neutral surface are stretched and the planes on the other side are squeezed. The displacement of any point on the neutral surface is $u_x^0 = 0$, $u_y^0 = 0$, but $u_z^0 = w(x, y)$. Remember since our first discussion on strain we write the displacement vector as a function of the initial, undeformed, position of every point in our domain. As the undeformed domain is a two-dimensional plane we choose our coordinate system such that this is the $x - y$ plane.

We shall denote the components of tensors in two-dimensions with roman indices, e.g., u_k , σ_{ij} , s_{lm} while preserving the greek indices for the three dimensional tensor u_α , $\sigma_{\alpha\beta}$, etc.

9.1.2 Boundary conditions

As we consider an infinitely large plane, there are two boundary conditions, one at the top surface of the thin three dimensional domain and one at the bottom surface. By (6.9) on both of these surface

$$\sigma_{xz} = 0, \quad \sigma_{yz} = 0, \quad \text{and} \quad \sigma_{zz} = 0 \quad (9.1a)$$

$$\implies \sigma_{zx} = 0, \quad \sigma_{zy} = 0, \quad (9.1b)$$

too because the stress tensor is symmetric. This assumes that there are no external surface forces acting on these two surfaces.

9.1.3 Elastic free energy of bent plate

Thus we have a situation where the stress is zero at $z = \pm h/2$, where h is the thickness of the plate. As h is very small, it is safe to assume that the same stresses are also zero inside the material. By inverting Hook's law, (6.16c),

$$Ys_{zx} = (1 + \sigma)\sigma_{zx} = 0 \quad (9.2a)$$

$$\implies \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} = 0 \quad (9.2b)$$

$$\implies \frac{\partial u_x}{\partial z} = -\frac{\partial u_z}{\partial x} \quad (9.2c)$$

As the bending is small and the plate is thin we can approximate the vertical displace of any point (u_z) by: displacement of the neutral plane:

$$u_z \approx u_z^0 = w \quad (9.3)$$

Equation 9.2 and (9.3) together implies

$$\frac{\partial u_x}{\partial z} = -\frac{\partial u_z}{\partial x} \approx -\frac{\partial w}{\partial x} \quad (9.4)$$

What could happen otherwise ? Think about the limit $h \rightarrow 0$.

As w is not a function of z , this equation can be integrated to obtain

$$u_x = -z \frac{\partial w}{\partial x} \quad \text{and} \quad u_y = -z \frac{\partial w}{\partial y} \quad (9.5)$$

The last equality follows from the symmetry between x and y . As we have u_x and u_y by virtue of the plate being thin, we can write down several components of the strain

$$s_{xx} = -z \frac{\partial^2 w}{\partial x^2} \quad s_{xy} = -z \frac{\partial^2 w}{\partial x \partial y} \quad (9.6a)$$

$$s_{yx} = -z \frac{\partial^2 w}{\partial x \partial y} \quad s_{yy} = -z \frac{\partial^2 w}{\partial y^2} \quad (9.6b)$$

Next we apply Hook's law to σ_{zz} which is zero by the boundary conditions:

$$\sigma_{zz} = 0 = \frac{Y}{1+\sigma} \left[s_{zz} + \frac{\sigma}{1-2\sigma} (s_{xx} + s_{yy} + s_{zz}) \right] \quad (9.7a)$$

$$\Rightarrow s_{zz} = -\frac{\sigma}{1-2\sigma} (s_{xx} + s_{yy}) = \frac{\sigma z}{1-2\sigma} \nabla_{\parallel}^2 w \quad (9.7b)$$

As we have now obtained all the components of the strain we can write the free energy of this thin three dimensional material as

The Laplacian operator in the two-dimensional plane

$$\nabla_{\parallel}^2 \equiv \partial_x^2 + \partial_y^2 \quad (9.8)$$

$$\mathcal{F}_{3d} = \int dx dy dz F(x, y, z) \quad (9.9a)$$

$$= \frac{Y}{2(1+\sigma)} \int dx dy dz \left(s_{\alpha\beta}^2 + \frac{\sigma}{1-2\sigma} s_{\mu\mu}^2 \right) \quad (9.9b)$$

$$= \frac{Y}{2(1+\sigma)} \int_{-h/2}^{h/2} z^2 dz \int dx dy (\cdot) \quad (9.9c)$$

$$\equiv \mathcal{F}_{\text{plate}} \quad (9.9d)$$

$$\text{where } \mathcal{F}_{\text{plate}} = \int dx dy \frac{Yh^3}{24(1-\sigma^2)} \left[(\nabla_{\parallel}^2 w)^2 + 2(1-\sigma) \left\{ \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right\} \right] \quad (9.9e)$$

Thus we have reduced the three dimensional elastic free energy to a free energy $\mathcal{F}_{\text{plate}}$ which is a function of only $w(x, y)$. From the free energy we now can write down the equations of elasticity for a thin two-dimensional sheet. This is not the same as writing equations for elasticity in two dimensions.

So far we have assumed that bending is small and the bending is *not accompanied* by stretching (of the neutral surface). These assumptions hold if a plane is bent into a cylinder but does not hold if it is bent into a spherical cap. We shall revisit this question later.

9.2 Elasticity of bent plates

To obtain the equation for elasticity of bent plates we now need to set the functional derivative of $\mathcal{F}_{\text{plate}}$ with respect to w to be zero.

$$\frac{\delta \mathcal{F}_{\text{plate}}}{\delta w} = 0 \quad (9.10)$$

This is the equivalent for Navier–Cauchy equation for plates that are *bent without stretching* and *by a small amount*. Quite magically, once you calculate the functional derivative and simplify the equation becomes

$$\nabla_{\parallel}^4 w = 0 \quad (9.13)$$

The bi-harmonic equation in two-dimensions. In presence of external forces this equation modifies to

$$\frac{\gamma h^3}{12(1 - \sigma^2)} \nabla_{\parallel}^4 w - P = 0 \quad (9.14)$$

where P is the surface force on the plate. If there are volume forces, e.g., gravity they must be integrated over the thickness of the plate to convert them to a surface force.

It is quite surprising to see that the term within the curly brackets in (9.9e) does not contribute ! You can consider this a somehow a magical cancellation of terms. But a more prudent decision is to try to see why this cancellation happens. This is what we try to do next.

9.3 Landau free-energy of bent plates

Let us try to directly write down the free energy of a bent plate just the way we did for solids. As we are not considering stretching at the moment, the only component of the displacement field that matters is $w(x, y)$ the displacement of the point with coordinate $(x, y, 0)$ to (x, y, w) . How will the free-energy depend on the field w ? Let us first try to construct the free-energy-density, F , as we had done for the 3D elastic body. We again set the zero of F to the state where $w = 0$. Then expand in a Taylor series

$$F = Aw + B_k \frac{\partial w}{\partial x_k} + C_{ij} \frac{\partial^2 w}{\partial x_i \partial y_j} + \text{h.o.t} \quad (9.15)$$

The first term is a translation, clearly F cannot change on translation, hence $A = 0$. The term $\frac{\partial w}{\partial x_k}$ denotes rotation, see Figure. 9.3. As we know, this cannot change F either, hence $B_k = 0$. The first non-zero contribution comes from the second derivative of w with respect to x and y . Usually, we write it in the following notation

$$F = C_{ij} \frac{\partial^2 w}{\partial x_i \partial y_j} + \text{h.o.t} \quad (9.16)$$

Consider a functional

$$\mathcal{A} = \int f(h, h', h'') dx \quad (9.11)$$

where $h(x)$ is a function of the independent variable x , and f is a function of h , h' and h'' – the symbol $'$ denotes derivative with respect to x . The functional derivative is given by

$$\frac{\delta \mathcal{A}}{\delta h} = \partial_x f - \frac{d}{dx} \left(\frac{\partial f}{\partial h'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial h''} \right) \quad (9.12)$$

So far we have considered the following transformations, translation along the vertical axis and rotation in the vertical planes. The free-energy density should also be unchanged (invariant) for rotation in the $x-y$ plane. Clearly, as I rotate my axis in the $x-y$ plane, individual entries of the matrix, with four entries $\frac{\partial^2 w}{\partial x_i \partial y_j}$, changes, and the second rank tensor C_{ij} also changes but their contracted product, which is a scalar, does not. Let us pause a moment and try to under-

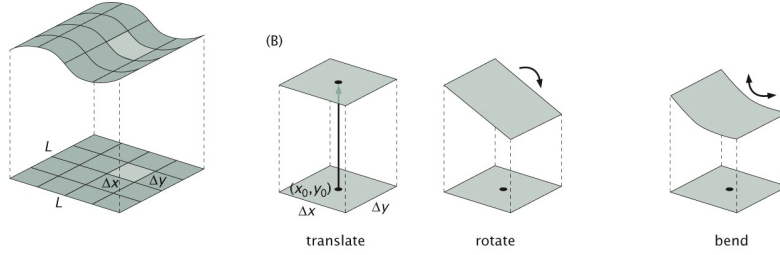


Figure 9.1: Possible deformations of a small element of area of a plate. Figure taken from *Physical Biology of the Cell*, Chapter 11.

stand the geometric meaning of what we have obtained. We are now looking at the matrix $\frac{\partial^2 w}{\partial x_i \partial y_j}$ where $w(x, y)$ is a height field above the $x-y$ plane. To keep matters simple let us consider a particular case where w is not a function of y but a function of only x . Then the only non-zero quantity in the matrix is $\frac{\partial^2 w}{\partial x^2}$ – the second derivative of the function $w(x)$. This derivative can have either sign, positive or negative, depending on whether the surface “curves up” or “curves down”. In the first case the Free energy is positive while in the second case it is negative. This cannot be the case for two reasons: (a) If this is the case, then F is not at a minimum; and (b) for a plate that is symmetric in top and bottom surface, the change in F should be the same irrespective of the sign of the second derivative, it should depend only on its magnitude. Hence (9.17) cannot be correct, the free energy density must depend on the square of the matrix $\frac{\partial^2 w}{\partial x_i \partial y_j}$.

$$F = C_{ijkl} \frac{\partial^2 w}{\partial x_i \partial y_j} \frac{\partial^2 w}{\partial x_k \partial y_l} + \text{h.o.t} \quad (9.17)$$

There is a neat way to write the right hand side of this equation. Notice that a 2×2 matrix has two invariants, its trace and its determinant. The invariant of a matrix mean exactly the same as invariant of a tensor. If we rotate our coordinate system then the elements of the matrix changes, but there are two properties of the matrix that does not. One, the sum of its eigenvalues, which is the trace. And two, the product of its eigenvalues, which is the determinant. Thus the free energy density, which depends on the matrix $\frac{\partial^2 w}{\partial x_i \partial y_j}$ must NOT be a function of its individual elements but its invariants. Further, the dependence cannot be linear but quadratic. The two quadratic

This does not hold if the plate is not made of isotropic material.

invariants are square of the trace and the determinant. Thus the free energy must have the form

$$F = C \left(\text{Tr} \frac{\partial^2 w}{\partial x_i \partial y_j} \right)^2 + D \det \left(\frac{\partial^2 w}{\partial x_i \partial y_j} \right) \quad (9.18)$$

Then the free energy would have the form

$$\mathcal{F}_{\text{plate}} = \int dx dy \left[B (\nabla_{\parallel}^2 w)^2 + \Lambda \left\{ \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right\} \right] \quad (9.19)$$

where B and Λ are two constant scalars which specify the material property of the plate. This expression exactly matches with the expression in (9.9) if we set

$$B = \frac{Yh^3}{24(1 - \sigma^2)} \quad (9.20)$$

and do not worry about Λ at the moment.

9.4 A detour in differential geometry

Let us take a quick detour in differential geometry. First consider a simple curve $f(x)$ as a function of x . The first derivative $f' = df/dx$ is the tangent of the angle the tangent to the curve makes with the x axis. The second derivative is related to the curvature by

$$\kappa = \frac{f''}{1 + (f')^2} \quad (9.21)$$

Now assume that the slopes are small, such that $(f')^2 \ll 1$. Then we can approximate the curvature by

$$\kappa \approx \frac{d^2 f}{dx^2} \quad (9.22)$$

The inverse of the curvature is the radius of curvature – the radius of the circle which is the best approximation to the curve at that point, see Figure. 9.2

For a surface, we can take a cut along the plane $y = 0$. This gives a cross-section of the surface, which is a plane curve given by $w(x, 0)$. If the elements of the strain, dw/dx are small then the curvature of the surface measured along the plane $y = 0$ is given by $d^2 w/dx^2$. Thus the matrix $\frac{\partial^2 w}{\partial x_i \partial y_j}$ contains all the information to calculate the curvature of the surface along any direction. The two eigenvalues of this matrix are called the two principal curvatures and their sum, the trace of the matrix, is called the mean curvature, see Figure. 9.4.

Thus with the approximation of small strain we can write

$$C_1 + C_2 = \text{Tr} \left(\frac{\partial^2 w}{\partial x_i \partial y_j} \right) \quad (9.23)$$

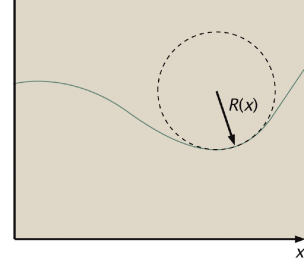


Figure 9.2: A plane curve and its radius of curvature.

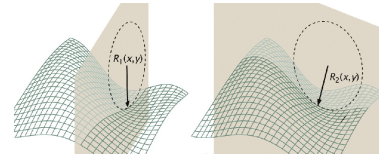


Figure 9.3: Two principal radii of curvature of a surface

where C_1 and C_2 are the two principal radii of curvature. The quantity, $H \equiv (C_1 + C_2)/2$ is called the mean curvature and the determinant of the matrix $\frac{\partial^2 w}{\partial x_i \partial y_j}$ is called the Gaussian curvature, K .

Now is the time to learn the statement of one of the most beautiful theorems of differential geometry, the Gauss–Bonnet theorem. Theorem states that the integral of the Gaussian curvature over a surface is a constant that depends only on the topology of the surface, i.e., how many holes it has.

9.5 Free energy in light of differential geometry

With our limited exposure to differential geometry, we can now write down the free energy as

$$\mathcal{F}_{\text{plate}} = \int dx dy \left[4BH^2 + \Lambda K \right] \quad (9.24)$$

The Gauss–Bonnet theorem states that the integral of the second term within the square bracket is a constant that depends only on the topology of the surface and its boundary. In all our deformations, we have never allowed for the surface to tear, i.e., we have kept the topology of the surface unchanged. Furthermore we keep the boundary condition the same. Hence this term is just a constant, it does not change on bending the surface. The free-energy of a surface due to bending is then given simply by

$$\mathcal{F}_{\text{bend}} = 4B \int dx dy H^2 \quad (9.25)$$

which explains why we can ignore the second term in the expression for free energy in (9.9e). For small strain

$$\nabla_{\parallel}^2 w = (\partial_x^2 + \partial_y^2)w = 2H \quad (9.26)$$

Remarkably the expression (9.25) holds even when the strain is not small, even when bending is large.

9.6 Looking back

Looking back, you will realise that the derivation of the free energy is actually quite clever and is quite generally applicable. Let me go through the steps again.

1. Assume small strain, use symmetries and the condition of thermodynamic equilibrium, to write down an expression for the free energy density. This expression, (9.18) is *coordinate dependent*.

2. By choosing a coordinate system we realise that the quantities appearing in the free energy are geometrically the mean and Gaussian curvatures of the surface. We can then rewrite the free energy again in a *coordinate independent* form, (9.25) but this time including the geometric insight.
3. Gauss–Bonnet theorem help us further simplify this expression.

The trick where we go through an explicitly coordinate dependent representation to eventually derive a coordinate independent form essentially uses the idea of gauge invariance. In general a complex surface in 3D must be represented by a set of 2D coordinate systems that are intrinsic to the surface, or by a parametric form. Such a 2D coordinate is not a Cartesian coordinate, the tensors in such a system are not the simple Cartesian tensors that we have been living with, but non-Cartesian tensor. There are many possible representations of the surface but the physics should be independent of the particular representation. This is key idea of gauge invariance. The choice of using a representation where the surface is a height field above a plane, is called the Monge representation or called choosing the Monge gauge.

For an excellent introduction to differential geometry of surfaces through tensor see [Aris, 1962]

9.6.1 Word of caution

We have the equation of mechanical equilibrium of a bent surface in (9.14), which is a fourth order PDE. We need to set boundary conditions to solve this PDE. This is where the term ΛK appears again. The precise statement of the Gauss-Bonnet theorem is as follows. For a two-dimensional manifold M with boundary ∂M

$$\int_M K dS + \oint_{\partial M} k_g d\ell = 2\pi\chi(M), \quad (9.27)$$

where k_g is the geodesic curvature of the boundary ∂M and $\chi(M)$ is the euler characteristic of M . Using this we can convert the second term in the free energy of a plate to a contour integral over the boundary. When we calculate the variation of the free energy with respect to w this contour integral contributes. The variation of the two-dimensional Laplacian in the free energy also gives rise to contour integrals that makes additional contribution. The sum of these contour integrals must be zero too. This gives quite nasty boundary conditions. If the edges of the plate are clamped or supported they simplify further. I am not going to consider this any further except directing you to the section 12 of the book on elasticity by Landau and Lifshitz [Landau and Lifshitz, 1970].

9.6.2 Bottom-up versus top-down

In the first part of the chapter we started with equations of elasticity in 3D and then zoomed out such that one direction of the 3D body became very thin. Thereby we derived an effective free energy for a two-dimensional surface. This could be called a bottom-up approach. This gave us an expression of the bending modulus (B) in terms of the elastic properties of the 3D material. In the second part of the chapter, we wrote down the free energy based on Landau like arguments which are very light on mathematical calculations. This can be called a top-down approach. However, this did not actually give us an expression for the bending modulus, it just showed that a bending modulus should exist. The argument of thermodynamic stability can again prove that the bending modulus is non-negative but not anything beyond that. This is the typical contrast between the two approaches.

9.6.3 Lipid Membranes

In biology the membrane of a cell is a very complex almost two dimensional object. The radius of typical cells are in micrometers, while the thickness of their membrane is about 5 nanometer. The membrane is made out of two layer of lipid, see Figure. 9.6.3.

Actual membrane contains several different kinds of lipids in different ratios. Cholesterol is a very important component. There are also proteins some of which are larger than the thickness of the bilayer – shown as a large cylinder in this sketch. The membrane is active, in the sense that it is driven by ATP from the cell – it can open pores, close them, selectively absorb small vesicles, etc. Some of the proteins are connected to an underlying network of polymers (actin polymers for most cells, spectrin for red blood cells). To understand such membranes we start from vesicles with a bilayer of lipids (typically a stable vesicle contains more than one type of lipid). The vesicles of the size of micrometers are called Giant Unilamellar Vesicles (GUV). A key difference between the membrane of such a vesicle and the membranes we are talking about is that such a membrane is not made by a thin three dimensional material but is actually a uniquely two dimensional material. The thinnest membrane we can make is single molecular sheet of graphite. This is also a uniquely two-dimensional material, not a thin three dimensional material. For these material, we cannot relate their bending modulus to the elastic modulus of any three dimensional elastic material.

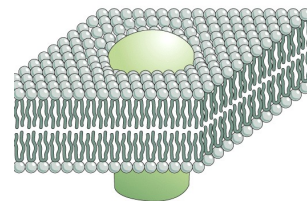


Figure 9.4: A sketch of a part of cell membrane. The lipids are molecules with a hydrophilic head and a hydrophobic tail. They form a bilayer. Actual membrane contains several different kinds of lipids in different ratios. Cholesterol is a very important component. There are also proteins some of which are larger than the thickness of the bilayer – shown as a large cylinder in this sketch. The membrane is active, in the sense that it is driven by ATP from the cell – it can open pores, close them, selectively absorb small vesicles, etc. Some of the proteins are connected to an underlying network of polymers (actin polymers for most cells, spectrin for red blood cells).

10 *Plates and Shells*

10.1 Bent and stretched plate

10.2 Force–distance relationship for plates

10.3 Shell

11 Elastic beams

Now we do down one more dimension and consider elasticity of rods. We use the words, rods, beams, filaments to mean the same thing. Filaments can be distorted quite easily into a variety of shapes. In general, they are curves in three dimensions. They appear in many problems from microscopic (microfilaments inside cells, flagella of cells, cantilever in an atomic force microscope) to large scales, e.g., bridges, pillars that hold the roof, etc.

There are three kinds of possible deformations in such objects. Stretch, bend and twist. Of which the stretch is quite simple. Quite often filaments are also unstretchable, i.e., they bend much more easily than they stretch. We do not consider stretching at all in what follows. Twist is a very important and very interesting phenomena. Particularly so because it brings in *handedness* into the problem. But in this elementary text we shall not consider twist. In this chapter, we shall focus on bending.

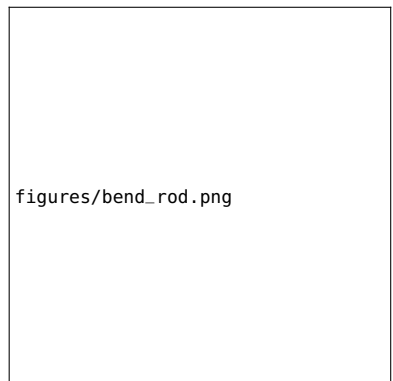
11.1 From 3D to 1D

The bending of rods is quite similar to bending of plates, except it is actually simpler. See a sketch in Fig. 11.1. Similar to what we talked about in plates, there is a neutral surface which neither stretch nor get compressed. The surfaces above the neutral surface get stretched while those below get compressed.

11.2 Solution of the beam equations

11.3 Buckling

11.4 Stability



figures/bend_rod.png

12 Flow of dry water: Principles

12.1 Fluid particle and its acceleration

Leaving elasticity, we now move on to flows. To describe a continuum we must now define a density and a velocity at every point in the fluid. We have been doing this already, without much thought on what we are actually doing. Let us now be somewhat finicky about this.

What we are doing here is setting up a *field theory*. In classical mechanics you have dealt with mechanics of single particle and also several particles interacting with each other. The number of degrees of freedom of such systems are finite, however larger. To specify a rigid body you have to specify its position (three numbers), and orientation (three angles) giving rise to six numbers. As the Newton's laws are second order in time, to specify an initial condition you need to specify both position (that includes angles) and velocities (that includes angular velocities). But to describe a continuum medium you need an infinite number of degrees of freedom. This is what a *field* is. One of the simplest example is a density field, $\rho(\mathbf{r})$. A scalar that is a function of position $\mathbf{r} = (x, y, z)$.

Velocity is also a field in exactly the same sense. We do not ascribe velocity to a mathematical point, but to a coarse-grained picture. A small volume in the fluid contains many molecules with each molecule moving in a different direction. The typical speed of these molecules are set by the local temperature. You may also know that the typical speed set by the temperature is of the same order as the speed of sound in this medium. But the velocity field we want to take about is the mean velocity of a large number of molecules. For almost all cases we consider in these lectures, this velocity much smaller than the speed of sound in the medium. Imagine a larger number of molecules each moving with quite a high speed in many different directions, but as far as their mean velocity is concerned the velocity of molecules moving in different directions cancel each other so we land up with a velocity that is quite small compared to the speed of individual molecules. This coarse-grained velocity itself

The idea of a density field should give you pause. Mathematically we can define it by

$$\rho(\mathbf{r}) = \lim_{\Delta V \rightarrow 0} \frac{\Delta m}{\Delta V} \quad (12.1)$$

Where ΔV is a small volume around the point $\mathbf{r}$ and Δm is the mass in that volume. But will this limit exist? Think about a glass of water. What is the density at one mathematical point? We know that the water is made out of molecules that are dancing around. So if we really look at the mathematical point there may or may not be a molecule at that point. So the density will be a highly fluctuating quantity. Even if we hit an atom, inside the atom there are electrons and a nucleus. You know that the nucleus occupies a very small position inside the atom and most of the atom is essentially empty. So as we go to finer and finer scales we start seeing the fundamentally discrete nature of matter. Hence to be able to define a density we need to work at scales much larger than the scale of molecules. So the limit should be understood from a *coarse-grained* view at a certain scale. How large should this scale be? It should be large enough to contain many molecules.

is different from one point in our medium to another.

For the simplest fluid imaginable, other than velocity and density there is another property that we must consider, its temperature. We first want to consider the simplest case where both temperature and density of the fluid is a constant but velocity is a function of space. This can be because the pressure is different in different parts of the fluid and also because of external forces acting on the fluid, e.g., gravity, which is the only external force we consider at the moment. What is the equation of motion of a small volume of fluid ? We had already deduced this in the previous lecture, Equation (??),

$$\rho \frac{dV_\alpha}{dt} = \partial_\beta \sigma_{\alpha\beta}, \quad (12.2)$$

which is true for both fluid and deformable solids. For fluid the stress tensor is given by, Equation (??) $\sigma_{\alpha\beta} = -p\delta_{\alpha\beta}$, where p is the pressure. Adding to this the gravity as an external force we eventually obtain

$$\rho \frac{dV_\alpha}{dt} = -\partial_\alpha p + \rho g_\alpha \quad (12.3)$$

which we now write in vector form

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla p + \rho \mathbf{g} \quad (12.4)$$

Remember this is the equation of a small parcel of fluid that itself moves in space. In classical fluid mechanics nomenclature this small parcel of fluid is called a “fluid particle”. Let me remind you how we derived Equation (12.2). The velocity $\mathbf{V}$ is the rate at which the position of a particle and the RHS of Equation (12.2) is calculated at its position. Let us assume that at time $t = 0$, the same fluid particle was at the position $\mathbf{r}^0$. The position of the fluid particle as a function of time is then given by $\mathbf{r}(t | \mathbf{r}^0, 0)$ where the initial position is used to label the particle. The instantaneous velocity and position

related by

$$\frac{d}{dt} \mathbf{r}(t | \mathbf{r}^0, 0) = \mathbf{V}(t | \mathbf{r}^0, 0), \quad (12.5)$$

which implies

$$\mathbf{r}(t | \mathbf{r}^0, 0) = \mathbf{r}^0 + \int_0^t \mathbf{V}(s | \mathbf{r}^0, 0) ds. \quad (12.6)$$

The displacement $\mathbf{r}(t | \mathbf{r}^0, 0) - \mathbf{r}^0$ is called the Lagrangian displacement and $\mathbf{V}(t | \mathbf{r}^0, 0)$ the Lagrangian velocity. Note that in our way of writing these equations, the velocity $\mathbf{V}$ is not a function of position, it is *not a field* in the usual sense of the word. It is rather a function of time. The space appears as a label that applies to the initial position of the fluid particle. So although we have derived an equation of motion for a fluid parcel we have not got what we expected – an

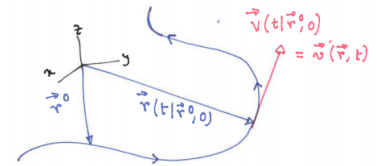


Figure 12.1: A sketch of the trajectory of a Lagrangian particle. The tangent to this trajectory at any instant gives the Lagrangian velocity as a function of time. This Lagrangian velocity is also the Eulerian velocity at the position of the Lagrangian particle.

A note for future references. In steady flows the Lagrangian particles do move along streamlines but not in general. So when I am drawing a picture of Lagrangian trajectories they have nothing to do with the streamlines.

equation of evolution of the velocity field. Let me denote the velocity field at a point $\mathbf{x}$ by $\mathbf{v}(\mathbf{x}, t)$. In fluid dynamics the velocity field is typically called *Eulerian velocity*.

Let us now revisit the acceleration term on the LHS of Equation (12.2).

$$\frac{d}{dt} \mathbf{V}(t | \mathbf{r}^0, 0) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{V}(t + \Delta t | \mathbf{r}^0, 0) - \mathbf{V}(t | \mathbf{r}^0, 0)}{\Delta t} \quad (12.7)$$

The **second term** in the numerator on the RHS of Equation (12.7) is related to the velocity field as follows

$$\mathbf{V}(t | \mathbf{r}^0, 0) = \mathbf{v}[\mathbf{r}(t | \mathbf{r}^0, 0), t] \quad (12.8)$$

where $\mathbf{r}$ is the Lagrangian displacement given in Equation (12.6). In a small interval of time t to $t + \Delta t$ the fluid parcel moves a distance

$$\Delta \mathbf{r} \equiv \mathbf{V}(t | \mathbf{r}^0, 0) \Delta t = \mathbf{v}(\mathbf{r}, t) \Delta t. \quad (12.9)$$

The **first term** on the numerator on the RHS of Equation (12.7) is then given by

$$\mathbf{V}(t + \Delta t | \mathbf{r}^0, 0) = \mathbf{v}(\mathbf{r} + \Delta \mathbf{r}, t + \Delta t) \quad (12.10)$$

Substituting Equation (12.10) and Equation (12.8) in Equation (12.7) and using Equation (12.9) we obtain

$$\frac{d}{dt} \mathbf{V}(t | \mathbf{r}^0, 0) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{v}(\mathbf{r} + \Delta \mathbf{r}, t + \Delta t) - \mathbf{v}(\mathbf{r}, t)}{\Delta t} \quad (12.11)$$

Now let us Taylor expand the first term in the numerator of Equation (12.11) in a multi-variable Taylor series

$$\begin{aligned} \mathbf{v}(\mathbf{r} + \Delta \mathbf{r}, t + \Delta t) &= \mathbf{v}(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) \\ &= \mathbf{v}(\mathbf{r}, t) + (\partial_x \mathbf{v}) \Delta x + (\partial_y \mathbf{v}) \Delta y + (\partial_z \mathbf{v}) \Delta z + (\partial_t \mathbf{v}) \Delta t \\ &= \mathbf{v}(\mathbf{r}, t) + [(\partial_x \mathbf{v}) v_x + (\partial_y \mathbf{v}) v_y + (\partial_z \mathbf{v}) v_z + (\partial_t \mathbf{v})] \Delta t \end{aligned} \quad (12.12)$$

where the last equality follows from Equation (12.9).

Compactifying the notation we can write Equation (12.12) as

$$\mathbf{v}(\mathbf{r} + \Delta \mathbf{r}, t + \Delta t) - \mathbf{v}(\mathbf{r}, t) = (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) \Delta t \quad (12.13)$$

Substituting this back in Equation (12.11) we get

$$\frac{d}{dt} \mathbf{V}(t | \mathbf{r}^0, 0) = \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \quad (12.14)$$

where the derivatives on the RHS are calculated at $\mathbf{x} = \mathbf{r}$ given by Equation (12.6). Putting the acceleration back in Equation (12.4) we obtain

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{g}. \quad (12.15)$$

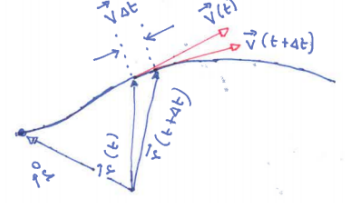


Figure 12.2: The difference between the Lagrangian velocity of the same Lagrangian particle at two different times is same as the difference is Eulerian velocity at two different space locations – the two different positions of the Lagrangian particle at two different times.

Allow me to remind you that we have assumed that the density ρ is a constant. The Equation (12.15) is the celebrated Euler equation. As you can see, this is a partial differential equation which is first order in space and time. To solve this we must specify boundary conditions.

The other property of this equation that you should note is that it is *non-linear* – the second term on the LHS of this equation. This term is also called the *advective derivative* of velocity. Both the beauty and the complexity of fluid mechanics owes its origin to this term. There is one crucial physical phenomenon that we have ignored so far in this lecture, that is viscosity. We know from experience that viscous forces exists. If I stir my coffee with a spoon and then stop the stirring, the flow slows down and eventually stops. This is because the layers close to the boundary slows down because of “friction” between the coffee and the cup. The inner layers of the fluid slows down because of the “friction” between the outer layers and the inner layers. We shall spend quite a large fraction of these lectures studying the effects of viscosity has on flows. The only exception is this lecture and the next.

In the ancient nomenclature the fluid that obeys the Euler equation is called *ideal* fluid. There is nothing “ideal” about such a fluid. Scientists have a tendency to take a word from everyday language and then to ascribe special meaning to it – a meaning may seem to be an abuse of the word. The word “ideal” does not mean something that normal fluid should aspire to be, it really means a hypothetical fluid. Let me give you a more familiar example. You know the *ideal* gas laws, although there are, in practice, no ideal gases. An ideal gas will remain a gas even when frozen to zero-degree Kelvin. There is no such gas. But real gases behave close enough to ideal gases over a certain region of their phase-diagram. The gas laws themselves are not hypothetical they are results of experiments. No flow can be described by the Euler equation, but in some special cases certain quantities calculated from the Euler equation may be a good approximation to those quantities in real flows. This statement may seem too abstract at the moment. We shall make it more precise near the end of this course. Following Feynman [Feynman et al., 1977], who himself borrowed the name from John Von Neuman, we shall call Equation (12.15) the equation for flow of dry water. A more mundane but accurate name is to call such flows *inviscid flows*.

Notation

Often we use a single symbol for the material derivative

$$D_t \equiv \partial_t + \mathbf{v} \cdot \nabla \quad (12.16)$$

You have learned the usual way of using vector identities to prove

$$\mathbf{v} \cdot \nabla \mathbf{v} = (\nabla \times \mathbf{v}) \times \mathbf{v} + \frac{1}{2} \nabla (\mathbf{v} \cdot \mathbf{v})$$

To show you the power of tensors, let me do it using what we have learned so far. Using the Levi-civita tensor the first term on the RHS can be written as:

$$\begin{aligned} & \epsilon_{\mu\alpha\nu} \epsilon_{\alpha\beta\gamma} (\partial_\beta v_\gamma) v_\nu \\ &= -\epsilon_{\alpha\mu\nu} \epsilon_{\alpha\beta\gamma} (\dots) \\ &= -(\delta_{\mu\beta} \delta_{\nu\gamma} - \delta_{\mu\gamma} \delta_{\nu\beta}) (\dots) \\ &= -\delta_{\mu\beta} \delta_{\nu\gamma} (\partial_\beta v_\gamma) v_\nu + \delta_{\mu\gamma} \delta_{\nu\beta} (\partial_\beta v_\gamma) v_\nu \\ &= -(\partial_\mu v_\gamma) v_\gamma + (\partial_\nu v_\mu) v_\nu \\ &= \mathbf{v} \cdot \nabla \mathbf{v} - \frac{1}{2} \nabla v^2 \end{aligned} \quad (12.17)$$

12.2 Bernoulli's equation

As the gravity can also be written as the gradient of a potential we can write Equation (12.15) as

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \left\{ \frac{p}{\rho} - \Psi \right\}. \quad (12.18)$$

remembering that ρ is a constant. Using the following vector identity

$$\mathbf{v} \cdot \nabla \mathbf{v} = (\nabla \times \mathbf{v}) \times \mathbf{v} + \frac{1}{2} \nabla (\mathbf{v} \cdot \mathbf{v}), \quad (12.19)$$

we can write Equation (12.18) as

$$\partial_t \mathbf{v} + (\nabla \times \mathbf{v}) \times \mathbf{v} = -\nabla \left\{ \frac{p}{\rho} + \Psi + \frac{1}{2} \mathbf{v}^2 \right\}. \quad (12.20)$$

Now consider steady flows, $\partial_t \mathbf{v} = 0$ and take a dot product of both sides of Equation (12.20) with the vector $\mathbf{v}$ to obtain:

$$\mathbf{v} \cdot \nabla \left\{ \frac{p}{\rho} + \Psi + \frac{1}{2} \mathbf{v}^2 \right\} = 0. \quad (12.21)$$

This is a remarkable result. To appreciate this first consider what the dot product means. As the flow is steady, at every point there is a constant velocity. I can think of a line, the tangent to which at every point gives the velocity at that point. Along such a line the quantity inside the curly braces in Equation (12.21) is zero ! Such lines are called *streamlines*.

The statement Equation (12.21) essentially is a statement of conservation of energy. Consider a fluid parcel of volume δV in a steady flow. Given its initial position it will move along the streamline that passes through that position. If we assume that energy is conserved – which is not the case as in practice viscous effects will make the particle loose energy – then the energy of this fluid parcel will be constant as it moves along a the streamline. The energy is precisely given by

$$p\delta V + \delta V\rho\Psi + \frac{1}{2}\rho\delta V\mathbf{v}^2.$$

Here the first term comes from the work done due to pressure, the second due to gravity and the last is the contribution from the kinetic energy. A more detailed version of this derivation can be found elsewhere, e.g., in Feynman [Feynman et al., 1977].

Now that we have arrived at the conservation of energy I shall now postpone a longer discussion of the Bernoulli's equation and talk about conservation laws in general for a moment.

Statics, steady flow and statistically steady flow:

In the last chapter, we studied hydrostatics, where $\mathbf{v} = 0$. Now we consider the cases where fluid actually flows, but it does not change with time $\partial_t \mathbf{v} = 0$, although $\mathbf{v}$ is a function of space. Can this truly happen ? Consider for example a flow in a pipe. If the pipe is not too big and the flow not too fast then you shall find that the velocity of water in the pipe is a constant with time. Although the velocity is different depending on how far you are from the walls of the pipe – at the wall the velocity is zero. But this is true only when the pressure difference across the pipe is a constant. This you can manage by attaching the inlet of the pipe to a water tank with is so large that within the time of your experiment the height of the water in the tank does not change by any appreciable amount. Again, it depends on length and time scales – how large the tank is and how long your experiment runs.

Much later while considering turbulent flow we shall consider *statistically* steady flows, in which case the flow velocity is not a constant but fluctuates a lot, nevertheless its statistics (mean, and all the higher moments) does not change with time.

12.3 Conservation Laws

I have stated many times that the laws of hydrodynamics are a particular example in a more general field, the theory of fields. Let me start the discussion on conservation laws in a more general and more abstract spirit.

Consider a system with large number of particles. Let us assume that each of these particles interact with every other particle. But all the interactions conserve certain quantities. For example they conserve the number of particles. This mean that under the interaction the particles may collide with each other but two of them are not allowed to merge together. Or one of them is not allowed to suddenly break up into two. So we reach a conservation law applicable to the whole volume:

Global Conservation Law. *Total number of particles in the volume V is a constant.*

If we consider any sub-volume then the conservation law as written above is not true any more. Because particle can enter or leave this volume. But we can modify the conservation law in a minor way to take care of this by saying

Local Conservation Law. *Total number of particles in any sub-volume δV plus the number of particles that enter minus the number of particles that leaves is a constant.*

Now we try to write a very similar law, instead of particles we consider a continuum with a density ρ . Instead of number of particles begin conserved we state that the total mass in the volume V must be conserved. A mathematical statement of this will be

$$\int_V dV \rho(\mathbf{r}) = \text{constant}, \quad (12.22)$$

or, in other words

$$\partial_t \int_V dV \rho(\mathbf{r}) = 0. \quad (12.23)$$

This is the global conservation law, which applies to the whole universe¹. But a local conservation law also holds:

$$\int_{\delta V} dV \rho(\mathbf{r}) + \left(\begin{array}{c} \text{mass that} \\ \text{enters } \delta V \end{array} \right) - \left(\begin{array}{c} \text{mass that} \\ \text{leaves } \delta V \end{array} \right) = \text{constant}, \quad (12.24)$$

which implies:

$$\partial_t \int_{\delta V} dV \rho(\mathbf{r}) + \left(\begin{array}{c} \text{rate at} \\ \text{which mass} \\ \text{enters } \delta V \end{array} \right) - \left(\begin{array}{c} \text{rate at} \\ \text{which mass} \\ \text{leaves } \delta V \end{array} \right) = 0, \quad (12.25)$$

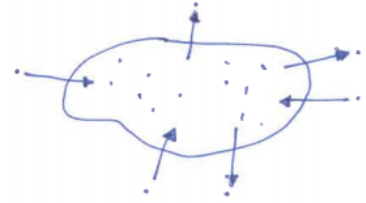


Figure 12.3: Particles moving in and out of a fixed volume in space.

¹ Actually this statement is not completely true. According to the laws of nature as we know today mass can disappear to be replaced by energy. As a concrete example an electron and positron can annihilate each other to give rise to light. But if we add the energy to the mass conservation law then we are correct again. In these lectures we shall stay away from both special and general relativity, hence mass itself is conserved.

Clearly mass must enter or leave the volume δV through its surface. So the second and third term on the LHS of Equation (12.25) are surface terms. They become clearer if we define a flux of matter, $\mathbf{J}$. The flux is a vector, such that the amount of matter that passes through a surface-element $\hat{n}dS$ in unit time is given by $\mathbf{J} \cdot \hat{n}dS$.

Clearly

$$\left(\text{rate at which mass enters } \delta V \right) - \left(\text{rate at which mass leaves } \delta V \right) = \oint_S \mathbf{J} \cdot \hat{n}dS, \quad (12.26)$$

where the surface S is a closed surface that bounds the volume δV . Putting everything together we obtain

$$\partial_t \int_{\delta V} dV \rho(\mathbf{r}) = - \oint_S \mathbf{J} \cdot \hat{n}dS \quad (12.27)$$

Now we apply Gauss's theorem to the RHS of this equation to obtain

$$\partial_t \int_{\delta V} dV \rho(\mathbf{r}) = - \int_{\delta V} \nabla \cdot \mathbf{J} dV \quad (12.28)$$

Let us consider a volume δV which itself does not change with time. In that case, the time derivative on the LHS can be moved inside the integral. Furthermore, this is true for any volume δV , hence we obtain

$$\boxed{\partial_t \rho + \nabla \cdot \mathbf{J} = 0.} \quad (12.29)$$

A wonderfully general relation that applies to any conserved "density" and its "flux". For example, in electrodynamics, electrical charge is a conserved density variable, in that case the vector $\mathbf{J}$ is the electrical current density.

The way it is written down now, Equation (12.29) is not of much use to fluid mechanics; we do not know what the flux $\mathbf{J}$ is. But what can move mass? Mass can move only if there is a flow. The flux of mass is then $\mathbf{J} = \rho \mathbf{v}$. Substituting this in (12.29) we get

$$\boxed{\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0.} \quad (12.30)$$

This is one of the central equations of fluid mechanics. It is known as the *continuity equation*. It is merely a statement of conservation of mass. So far in this chapter we have assumed that the density is a constant. But (12.30) is more general, it is true irrespective of whether density is constant or not.

If we apply (12.30) to flows where the density is indeed a constant, for example for water under most engineering and geophysical situations, we obtain

$$\nabla \cdot \mathbf{v} = 0. \quad (12.31)$$

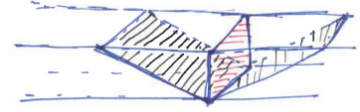


Figure 12.4: The flux of a quantity through a surface $\hat{n}dS$ is given by the product of the flux vector $\mathbf{J} \cdot \hat{n}dS$. All the three surfaces sketched above have the same flux.

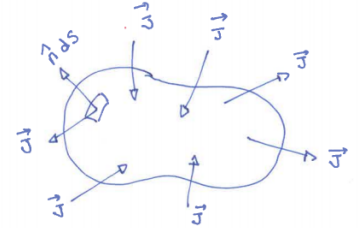


Figure 12.5: The flux of through the surface which encloses a volume is an integral over the surface.

A fluid that obeys this, i.e., a fluid whose density is a constant, is called an incompressible fluid. We shall sometime call (12.31) as the *incompressibility constraint*.

Exercise 3

1. Vorticity equation

Starting from Equation (12.20) show that the Euler equation can also be written in the form

$$\partial_t \boldsymbol{\omega} + \nabla \times (\boldsymbol{\omega} \times \mathbf{v}) = 0. \quad (12.32)$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$.

2. Incompressibility

1. Show that for incompressible flows, $\nabla \cdot \mathbf{v} = 0$, in the absence of any external force including gravity, the pressure and velocity are related by the following relation

$$\nabla^2 p = -\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}). \quad (12.33)$$

2. Can we have sound waves in a truly incompressible fluid ?

13 Flow of Dry Water: consequences

13.1 Dos and Don'ts of the Bernoulli's equation

The Bernoulli's equation is a much used and often abused concept in fluid mechanics. To see its power and its limitations let us work out a few examples.

13.1.1 Flow out of a hole

Take a water tank with a hole at its side near the bottom. If you open the hole what will be the velocity of the water coming out ? You can work this out using Bernoulli's theorem. Consider a streamline from the surface of the water in the tank down to the hole and apply Bernoulli's equation to it. Let us take the zero of the gravitational potential Ψ at the surface of the water in the tank where the velocity is zero. As the pressure at the two ends of the streamline is give by the atmospheric pressure the equation looks like:

$$p_o = p_o + \frac{1}{2}v_{out}^2 - gh \quad (13.1)$$

which implies $v_{out} = \sqrt{2gh}$. This is known as Toricelli's law. Note that you cannot get the rate at which water leaves the tank by just multiplying this velocity by the area of the hole. To quote Feynman ¹ " The fluid velocities as the jet leaves the hole are not all parallel to each other but have components inward toward the center of the stream – the jet is converging. After the jet has gone a little way, the contraction stops and the velocities do become parallel. So the total flow is the velocity times the area at that point. In fact, if we have a discharge opening which is just a round hole with a sharp edge, the jet contracts to 62 percent of the area of the hole. The reduced effective area of the discharge varies for different shapes of discharge tubes, and experimental contractions are available as tables of efflux coefficients." Clearly, everything that I have said above assumes that the tank is large enough or the hole is small enough such that when the water flows out of it the height of the water in the tank changes slowly compared to the rate at which water flows out of

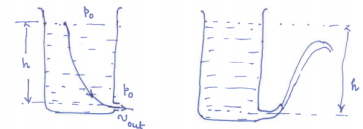


Figure 13.1: Flow out of a hole at the bottom of a tank

¹Richard P Feynman, Robert B Leighton, and Matthew Sands. The Feynman Lectures on Physics: Mainly Electromagnetism and Matter, Vol. 2. Addison-Wesley, Reading. reprinted, 1977

the tank. There are however bigger conceptual problem in applying Bernoulli's equation to this problem. If instead of letting the water out we added a discharge tube such that the water flowed upwards then if Bernoulli's equation were correct water would rise to the level of water in the tank itself. This simple experiment, which I am not going to do in class, shows that water always falls short. This is because of the loss of energy due to viscous effects.

13.1.2 Bernoulli levitation

A nice demonstration of the the Bernoulli's equation is given by Waltham et al². A pump forces air through a narrow horizontal channel as shown in Figure. 13.2. As the velocity is increases the pressure decreases, this make the atmospheric pressure outside lift up the bottom plate. In the absence of the necessary equipment you can see a theorist's version of this experiment as was suggested by Sommerfeld³. Take a piece of light paper on a table. I have got the best effects by using a square of toilet paper. Then blow at it through a small slit created between two of your fingers. With a little bit of practice you can lift the paper up. It works better the lighter the paper is. If by some chance you have papers people used to roll up cigarettes you may have very good results. The flow comes down through the slit and then moves horizontal.

13.1.3 Can Bernoulli fly aeroplanes ?

It is now natural to ask does the Bernoulli's equation describes, at least qualitatively, how aeroplanes get their lift ? There are many places in the Internet, including webpages hosted by NASA and MIT that says yes. In truth the answer in no. It will take some time to explain. For the first half of this discussion I shall follow closely an article⁴ by Holger Babinsky.

The popular argument goes as follows: The typical shape of a wing it is curved on one side and flat on the other. The distance from the point at the front on the wind to its tail is greater if you follow the upper edge and less if you follow the lower edge. So you may argue that two fluid particles that follow two different streamlines – one just above and one just below the wing – reaches the tail at the same time. Hence the one above goes at faster speed. According to Bernoulli's equation faster speed gives rise to lower pressure, hence there will appear lower pressure on the top of the wind and higher on the bottom. This explains the lift. This explanation is wrong on several counts

1. First, Bernoulli's equation applies to individual streamlines, it

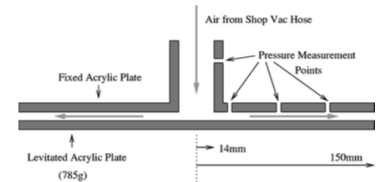


Figure 13.2: A sketch of the experiment by Waltham et al.

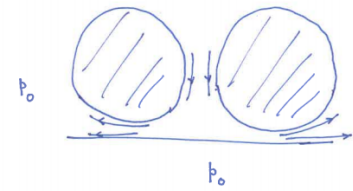


Figure 13.3: A sketch of blowing at a piece of paper through your fingers.

²Chris Waltham, Sarah Bendall, and Andrzej Kotlicki. Bernoulli levitation. *American Journal of Physics*, 71(2): 176–179, 2003

³

⁴Holger Babinsky. How do wings work? *Physics Education*, 38(6):497, 2003

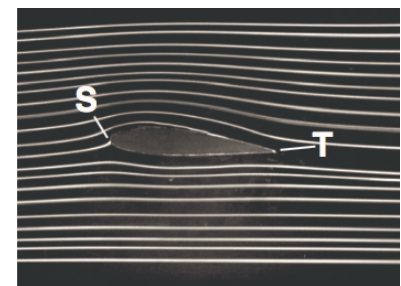


Figure 13.4: Streamlines about the cross-section of a wing visualized with smoke. Figure taken from the paper by Babinsky.

does not allow you to compare two different streamlines with each other. So the application of Bernoulli's equation in the manner just described is essentially wrong.

2. Second, you should ask why should the two fluid particles meet at the end of the wing ? I do not know why. And real life observation show that they do not, see for example the movie at <http://iopscience.iop.org/article/10.1088/0031-9120/38/6/001/data> that accompanies the paper by Babinsky. In Figure. 2 we show three snapshots from the movie.

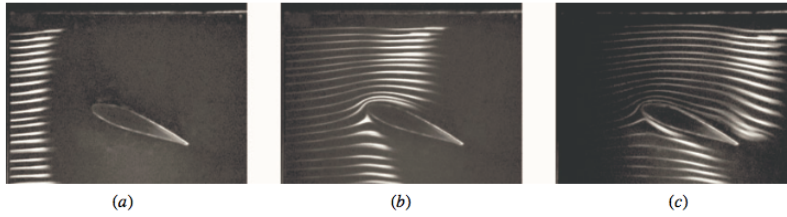


Figure 13.5: Three snapshots of smoke particles flowing along a wing. Notice how the particles above reach the end of the wing before the particles at the bottom. The statement that they should reach the trailing end of the wing together is wrong. The figure is taken from the paper by Babinsky.

3. You can generate a lift even in the case where two sides have the same length. A typical example is that of a sail.

So, how does the aeroplane fly then ? At this moment I am not giving you an answer. Later in the course we hope to calculate the drag and the lift forces on a wing. It will then become clear exactly how and why planes fly. Nevertheless, not that in all our discussions about wing and flights so far we have ignored the role of viscosity. In this particular case viscosity plays a crucial role. The speed of the flow at any boundary must be zero. This condition is imposed by the viscosity. The interaction between the wing and the air is through viscous effects. You cannot understand flights unless you understand viscous effects. The Euler equation and the Bernoulli equation which is a consequence of it does not contain viscous effects, hence they cannot describe either drag or lift.

There are parts of Internet that debates whether Newton's laws or Bernoulli's equations are responsible for the lift. This question itself is meaningless because Bernoulli's equations are consequences of Newton's laws. Whatever we do in classical mechanics is a consequence of Newton's laws.

13.2 A solution of Euler equation

The steady Euler equation, in the absence of gravity, can be written in the following way:

$$(\nabla \times \mathbf{v}) \times \mathbf{v} = -\nabla \left\{ \frac{p}{\rho} + \frac{1}{2}v^2 \right\} \quad (13.2)$$

This is clearly a non-linear equation, hence in general there can be more than one solution. The form of the equation suggests that there can be a solution such that

There is a systematic way of solving the eigenvalue problem in Equation (13.6). By taking another curl and using $\nabla \cdot \mathbf{v} = 0$ we obtain

$$\nabla^2 \mathbf{v} + \Lambda^2 \mathbf{v} = 0, \quad (13.3)$$

which is the vector wave-equation. This is solved by

$$\mathbf{v} = \frac{1}{\Lambda} \nabla \times \nabla \times (\hat{n}\psi) + \nabla \times (\hat{n}\psi) \quad (13.4)$$

$$\nabla \times \mathbf{v} = \Lambda \mathbf{v} \quad (13.6)$$

where Λ is a constant. Clearly this satisfies the incompressibility constraint

$$\nabla \cdot \mathbf{v} = 0. \quad (13.7)$$

(How will you show this ?) If you substitute Equation (13.6) in Equation (13.2) the LHS side is zero. What about the RHS ? Can you show that it is zero too ? A different way to show that Equation (13.6) gives one solution of the Euler equation is to use the vorticity equation that you derived in Exercise 4, Problem 1, and consider the steady case $\partial_t \omega = 0$, in which case we end up with

$$\nabla \times (\omega \times \mathbf{v}) = 0,$$

which is clearly satisfied by Equation (13.6). So we have reduced the problem of finding a solution to the steady Euler equation to a linear problem, that of finding a solution to the linear equation Equation (13.6)⁵. There is a wonderfully symmetric solution given by the following expression

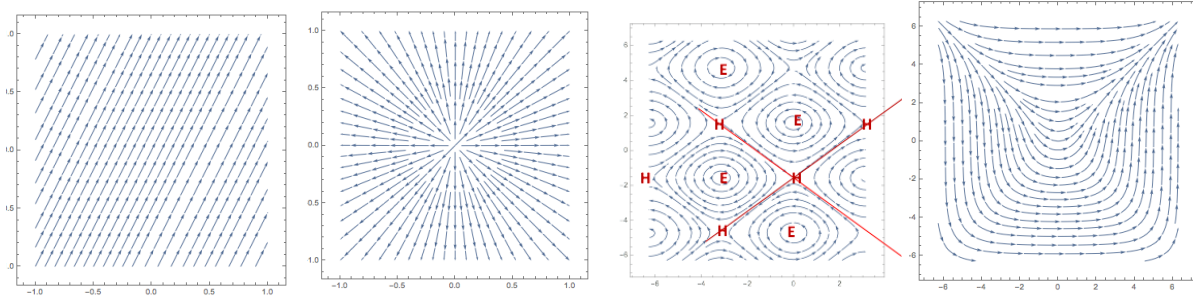
$$v_x(y, z) = A \sin \Lambda z + C \cos \Lambda y \quad (13.8a)$$

$$v_y(x, z) = B \sin \Lambda x + A \cos \Lambda z \quad (13.8b)$$

$$v_z(x, y) = C \sin \Lambda y + B \cos \Lambda x \quad (13.8c)$$

This is called the ABC (Arnold-Beltrami-Childress) flow. You can

⁵You need to be a little cautious here. Even if we manage to solve Equation (13.6) we get only *one* solution of the Euler equation. There may be many more.



check this by direct substitution. There are intriguing properties of these functions, some of which will be explored in the exercises.

13.3 Streamlines

Let us now revisit the streamlines. We have a velocity field $\mathbf{v}(x, y, z)$ in space that is constant in time. We want to find the streamlines. Let us first consider the problem in two dimensions. At any point x, y , the velocity has two components $v_x(x, y)$ and $v_y(x, y)$, which are

Figure 13.6: Few examples of streamlines. From left to right: $v_x = 1, v_y = 2$, $v_x = x, v_y = y$, $v_x = \cos(y), v_y = \sin(x)$ and $v_x = \cosh(y), v_y = \sinh(x)$. The second plot has non-zero divergence at the origin – a source. This cannot be the streamlines of an incompressible flow. The third plot is particularly interesting. The lines that separate the patches of circles from one another are called separatrices. The center of the circle and the point where two separatrices cross are *fixed points* of this system – points where the velocities are zero, also called stagnation points in fluid mechanics. There are two types of them – ones around which the flow looks elliptical, these are marked by **E** and are called elliptic points. The others around which the flow looks hyperbolic, these are marked by **H** and are called hyperbolic points. Plotted using the `StreamPlot` command of Mathematica.

both function of space. Let us write them as $F(x, y)$ and $G(x, y)$. To parametric equation for the streamlines can be obtained by solving

$$\frac{dx}{dt} = v_x(x, y) = F(x, y) \quad (13.9a)$$

$$\frac{dy}{dt} = v_y(x, y) = G(x, y) \quad (13.9b)$$

These equations can be considered a two coupled ordinary differential equations (ODEs). The functions F and G can in general be nonlinear. The problem is then an initial value problem with the initial position of the fluid particle as the point where the streamline starts. You can think about several simple cases where these equations can be easily solved.

But we are not going to discuss those special cases now – they can be found in the appendix of this chapter. I have plotted several typical examples of streamlines in Figure. 13.2 in two dimensions.

13.3.1 The general case

To think about the general case think about the problem in the following way. Equation (13.9) forms a two dimensional dynamical system. There are certain properties that F and G must satisfy, in particular, they must be such that the Bernoulli's equation holds. That implies that there is one *integral of motion* – the energy. This sets certain constraint on the system. In particular, in two dimension, one constraint implies that the motion must remain on a curve. This means that there exists a transformation (in general nonlinear), from x, y to ξ, η such that the motion happens along the curve $\xi = \text{constant}$. This means that Equation (13.9) is an *integrable system*. The other coordinate η must do one of the two things : (a) it can go away to infinity (open streamlines) (b) it must be a periodic variable, like an angle.

13.3.2 In three dimensions

Why did I spend so much time in this section on a problem that seem pretty obvious ? Thinking in a simple way, in a steady flow fluid particles follow streamlines. As one fluid particle cannot go in two different paths and must go along exactly one path, the solution to Equation (13.9) must exist and must be unique. Actually, irrespective of the number of dimensions, it should hold in three dimensions too. And indeed it does, streamlines in three dimensions are unique and they exist but they are not always *integrable*. Revisiting the argument about an integral of motion, when we are in three dimensions we have three equations with one constraint (the integral of motion). This confine a streamline to a surface, not a line. This fact has a

An integral of motion of a dynamical system is something that remains constant as the system evolves.

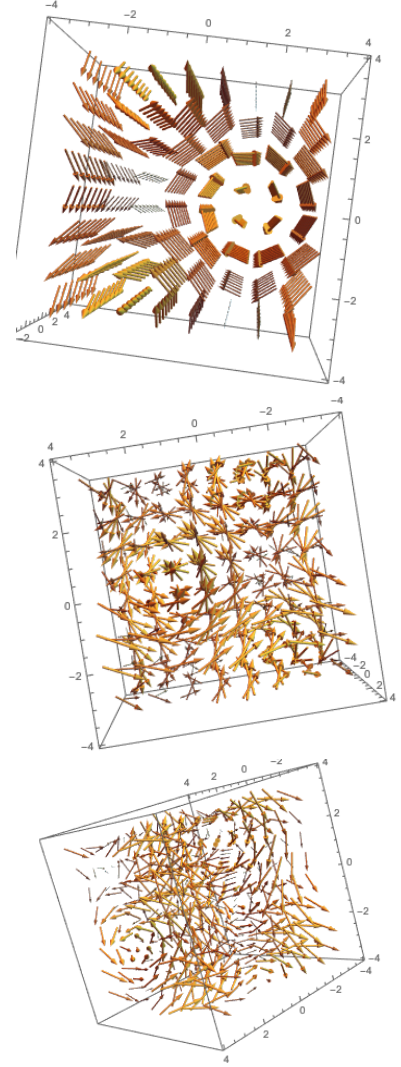


Figure 13.7: Three graphical representations of the ABC flow. In all the three $\Lambda = 1$. In the top one $C = 0$. The second and the third are two different views of vector plot of the flow

major implication. For example, the streamline can now cover this surface *ergodically* – it can come as visit almost every point on this surface, like wrapping a thread of wool on a ball – of course without ever intersecting itself. Secondly, two streamlines that initially are very close to each other can go away from each other *very fast* – in particular exponentially fast. Let us start with two points in x, y, z given by $\mathbf{r}^0$ and $\mathbf{r}^0 + \Delta\mathbf{r}$. As we follow the two streamlines that evolves from these two points we may find that the distance $\Delta\mathbf{r}$ grows exponentially fast in time, e.g., as $\exp(\lambda t)$ where λ is positive. This is a typical feature of *chaotic* systems – sensitive dependence on initial conditions. And the exponent λ is called the largest Lyapunov exponent. So the streamlines in three dimensions can be chaotic – not so in two dimensions. An example of chaotic streamlines is actually the ABC flow (13.8) for $A = B = C = 1$. It is not chaotic if any of the three constants are zero. The beauty and complexity of this problem has been explored in several papers, I suggest you start with the one by Dombre and his collaborators ⁶.

13.4 Back to conservation laws

Let us now revisit the conservation law we derived in the last lecture. We noted that for any conserved density variable ρ we can write down a conservation law of the general form

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0 \quad (13.10)$$

We have already applied this to mass and obtained the continuity equation. Let us try to apply this to the next conserved quantity momentum. The momentum density function $\mathbf{P} = \rho\mathbf{v}$. But momentum is not conserved in the same way that mass is. If we consider a small volume of the fluid – this time not a fluid particle but a fixed volume δV in space – then the momentum of this volume changes in two ways. One, the one we are already familiar with is the divergence of all stresses acting on this volume. As the divergence of stresses constitute a force they contribute to the rate-of-change of momentum to this volume. So the conservation equation now looks like

$$\partial_t P_\alpha - \partial_\beta \sigma_{\alpha\beta} + \nabla \cdot (\text{flux of } P_\alpha) = 0 \quad (13.11)$$

What is the flux vector of a component of momentum? Remember, the flux of any quantity due to the flow is the same quantity multiplied by the velocity. The flux of a scalar density is a first rank tensor – a vector. The flux of a vector quantity is, you guessed it, a second rank tensor. The flux of momentum is given by

$$\Pi_{\alpha\beta} = v_\beta P_\alpha = \rho v_\alpha v_\beta \quad (13.12)$$

⁶T. Dombre, U. Frisch, M. Henon, J. M. Greene, and A. M. Soward. Chaotic streamlines in the ABC flows. *Journal of Fluid Mechanics*, 167:353–391, June 1986. DOI: 10.1017/S0022112086002859

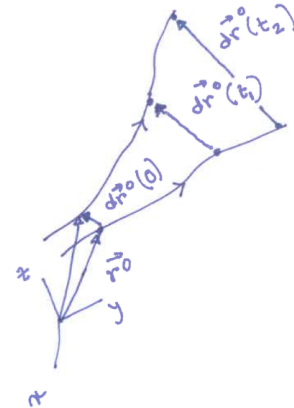


Figure 13.8: Two streamlines that are very close to each other near $\mathbf{r}^0$ diverges away from each other exponentially fast. The exponent give the largest Lyapunov exponent. This is a typical signature of chaos.

Substituting this back in Equation (13.11) we obtain

$$\partial_t(\rho v_\alpha) + \partial_\beta [\rho v_\alpha v_\beta - \sigma_{\alpha\beta}] = 0 \quad (13.13)$$

Can you see that this is the Euler equation reborn ?

Exercise 4

1. Helicity

The helicity density, at a point in a flow is given by

$$h = \boldsymbol{\omega} \cdot \mathbf{v} \quad (13.14)$$

where the flow velocity is given by $\mathbf{v}$ and the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. Calculate the total helicity $\mathcal{H}$ of the ABC flow defined by

$$\mathcal{H} = \int_{-\pi}^{\pi} dx \int_{-\pi}^{\pi} dy \int_{-\pi}^{\pi} dz (\boldsymbol{\omega} \cdot \mathbf{v}), \quad (13.15)$$

for $A = B = C = 1$.

2. Using the continuity equation, show that Equation (13.13) is the Euler equation.

3. The Venturi tube

A sketch of the Venturi tube is given in Figure. 13.9. Using the continuity equation and the Bernoulli's equation show that the flowrate through the tube is given by

$$Q = \frac{A_2}{\sqrt{1 - (A_2/A_1)^2}} \sqrt{2gh} \quad (13.16)$$

where g is the gravitational acceleration and rest of the symbols are explained in the caption of Figure. 13.9. In practice the formula is wrong; the RHS must be multiplied by a “fudge factor”, C which must be determined experimentally.

13.5 Measurement of flow velocity

Unfortunately this course is a totally theoretical one, while fluid dynamics is a subject that is intimately connected with experiments. I have neither the skill nor the time to go into measurement of flow properties in any detail. Let me just touch on few methods that are commonly used to measure flow velocities.

13.5.1 Venturi tube and Pitot tube

Notice that if the fluid passes through a constriction its velocity should increase (why ?) and this means that locally the pressure will

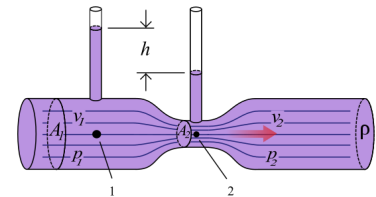


Figure 13.9: A sketch of Venturi tube. The flowrate through the tube is given by Q , in Equation (13.16). The area of the wider part of the tube is A_1 and the area of the narrower part is A_2 . The difference in height of the pressure heads is h .

decrease. By measuring this drop in pressure through the Bernoulli's equation you can measure the flow velocity.

A very clever generalisation of this idea is implemented in Pitot tube. In its simplest avatar it is just a bent tube. If you put it in a river (according to rumours Pitot used the Seine) the level of water that rises in the tube is determined by the balance between the pressure in the flow and the atmospheric pressure. The pressure in turn depends on the flow velocity and thus the level up to which the water rises depends on the flow velocity. If you put down a bent tube in a river and turn it around the height up to which water rises in the tube changes depending on whether the hole faces the flow direction or not. A more sophisticated version is a design where there are two holes one at the end and one at the side. The difference in the pressure head, h , in these two channels gives the flow velocity as $v = C\sqrt{2gH}$. Notice that it must be multiplied by a "fudge factor" of C called the coefficient of velocity. We faced similar factors while dealing with flow out of a hole too.

13.5.2 Hot air anemometer

The fact that air cools down a hot object can be used to measure the speed of air. This is the principle of a hot wire anemometer. The idea is that you control the heating of a wire by an electrical circuit to hold its temperature constant. This implies that the rate at which electrical energy is dissipated is the same rate at which it is carried away by the wind. To quote from Encyclopedia Britannica "In the most common type of hot-wire anemometer, the constant-temperature type, power is increased to maintain a constant wire temperature. The input power to the hot wire is the n a measure of airspeed, and a meter in the electrical circuit of the hot wire can be calibrated to indicate airspeed. This device is useful for very low airspeeds, below about 5 miles (8 km) per hour." These are most commonly used in wind tunnels and aboard aircrafts. I enthusiastically recommend the Wikipedia article on anemometers: <https://en.wikipedia.org/wiki/Anemometer>.

13.5.3 Particle image velocimetry

This is one of the modern methods of measuring flow velocities that has revolutionised research in fluid mechanics. The idea is that many small spheres with almost the same density as the fluid itself is released in the fluid. These are then tracked by very fast cameras. By comparing the position of the particles from one frame to another their velocities are measured. What you get is essentially very close to the Lagrangian velocity of many particles which allows you to

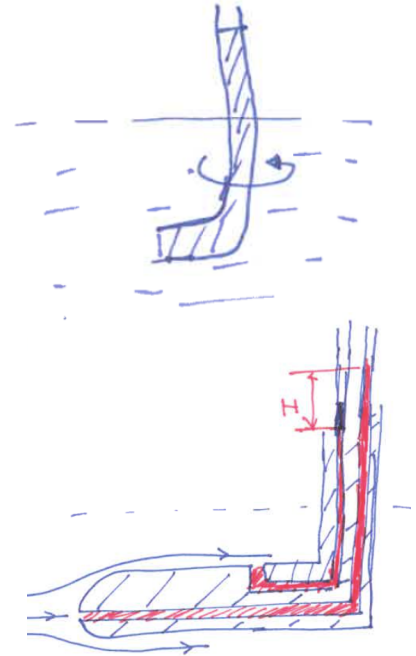


Figure 13.10: Sketch of the Pitot tube.

map out the Eulerian velocity field too at that instant. You get as close to Lagrangian velocity as you can match the density of the particles with that of the fluid. The smaller the particles are, more accurate are the measurements.

13.6 Kelvin's theorem

We are now at the end of our discussion of flow of “dry water”. There is one very important result that I need to discuss. This states that

Theorem. *In inviscid fluid the flux of vorticity through an open Lagrangian surface remains constant in time.*

Before proving it let us try to understand its content. First note that it says “Lagrangian surface”, what does that mean? It means the following: at time $t = 0$ you mark a surface, i.e., you mark every point on the surface; then calculate the flux of vorticity through this surface. Then as time evolves that Lagrangian particles move elsewhere, as they do the surface changes but the flux of vorticity through this “new” surface is the same as it was through the “old” surface. So mathematically we restate the theorem as

Theorem. *In inviscid fluid*

$$D_t \int_S \boldsymbol{\omega} \cdot \hat{\mathbf{n}} dS = 0 \quad (13.17)$$

where $D_t \equiv \partial_t + \mathbf{v} \cdot \nabla$ is the material derivative.

Note next that the theorem say, *open surface*. From Stokes theorem, you know that the flux of any vector field through an open surface is equal to its line integral over the line that bounds the surface. So an alternative statement of the Kelvin's theorem is

Theorem. *In inviscid fluid*

$$D_t \oint_{\Gamma} \mathbf{v} \cdot d\boldsymbol{\ell} = 0 \quad (13.18)$$

where $D_t \equiv \partial_t + \mathbf{v} \cdot \nabla$ is the material derivative.

The integral of vorticity along a closed line is called circulation

$$C = \oint_{\Gamma} \mathbf{v} \cdot d\boldsymbol{\ell}. \quad (13.19)$$

With this definition we can restate Kelvin's theorem as

Theorem. *In inviscid fluid the circulation of any closed Lagrangian line remains constant in time.*

Note next that this theorem is true only for inviscid fluid, i.e., dry water. So far in this chapter we have seen that most conclusion that we draw from our theory of dry water are quantitatively wrong – most of the time we have to include an arbitrary factor to fix it to match experiment. Do we need to do something similar for the Kelvin’s theorem when we deal with real fluids ? I do not/can not answer this question at this point, but just want to raise it.

13.6.1 Potential flows

Note a particularly interesting consequence of this theorem. If to begin with vorticity was zero everywhere in a domain then in an inviscid fluid it will remain so at all future times. This makes it particularly interesting to consider the case of *irrotational flows* where

$$\nabla \times \mathbf{v} = 0 \quad \text{everywhere.} \quad (13.20)$$

For irrotational and incompressible fluid the equation of motion are given by

$$\nabla \cdot \mathbf{v} = 0 \quad \text{and} \quad \nabla \times \mathbf{v} = 0. \quad (13.21)$$

As the curl of velocity is zero, we can define a potential function ψ such that

$$\mathbf{v} = -\nabla\psi \quad (13.22)$$

because the curl of any gradient is zero. The potential for velocity then satisfies the equation

$$\nabla^2\psi = 0. \quad (13.23)$$

This is called Laplace’s equation. Remarkably, we have now reduced the problem to solving a partial differential equation for a scalar field. This equation is also called the potential equation and the velocity that is a solution of such an equation the *potential flow*. If this were a traditional fluid mechanics course, we shall now spend quite a lot of time solving the potential equation in all its glory. The necessary mathematics is beautiful and important, but is not of much use in reality because real flow are not potential at all. Except in one very important case. Often the real flow about a real wing can be thought about as two flows, one outer – somewhat away from the wing – that is indeed irrotational and one inner – quite close to the wing (boundary layer) – that is controlled by the viscous effects. To find a complete solution these two solutions must be *matched* with each other. The solution of potential flow problems then do become important. I shall come back to the potential flows when I need them, but now I shall concentrate on the viscous effects starting with the next chapter.

A word of warning. If we assume an initially irrotational flow everywhere in a real fluid you are the flow will not necessarily remain irrotational in future. Vorticity is going to be generated, due to viscous effects mostly near the boundary of the flow. The vorticity thus generated will diffuse into rest of the flow. We shall spend some time discussing this mathematically few chapters later.

The Equation (13.21) is the same equation satisfied by a static electric field in a domain where the charges are confined only the boundary of the domain. For example inside a capacitor. Or inside a metal box whose different faces are raised to different potential. Equation (13.23) is then the equation satisfied by the electrostatic potential. If you have had a typical undergraduate course on electromagnetism, something at the level of Griffith’s book [Griffiths, 1962] you already know quite a lot about this problem.

Summary

Theorem. *The equation of motion of a fluid with constant density, flowing under gravity, is given by the Euler equation*

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{g},$$

with the incompressibility constraint

$$\nabla \cdot \mathbf{v} = 0,$$

if we ignore the viscous forces. Here $\mathbf{v}$ is the velocity, p the pressure, ρ the density, and $\mathbf{g}$ the gravity.

From this we can derive the Bernoulli's theorem which states that

$$\mathbf{v} \cdot \left\{ \frac{\mathbf{p}}{\rho} + \Psi + \frac{1}{2} \mathbf{v}^2 \right\} = 0$$

in a steady flow. Here $\mathbf{g} = -\nabla \Psi$, i.e., Ψ is the gravitational potential.

Theorem. *If ρ is a conserved density variable and $\mathbf{J}$ is its current then they together satisfies the local conservation law*

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0$$

A consequence of this theorem and the fact that mass is conserved is the continuity equation

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{13.24}$$

where ρ is the mass density and $\mathbf{v}$ is the velocity of the flow. This is true irrespective of whether we consider incompressible or compressible fluids.

14 Viscosity

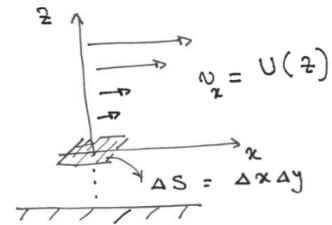
14.1 A microscopic picture

Now we get in to real fluids. You know the viscosity exists, for example, if I stop stirring my coffee it stops after a little while. Intuitively this looks like forces like friction but not between solids but between two layers of a liquid. Let us try to mathematically describe this physical phenomenon. We know already the liquids cannot sustain shear stress, they flow. Consider the simplest case where a liquid moves in the x direction with a velocity $v_x = U$ that is a function of the vertical coordinate z . If there were no relative motion, there would be no viscous forces. So to the first approximation we consider the viscous forces to be proportional to the gradients of velocity. Of course, we should talk about viscous stresses, not forces. Consider a small element of area $\Delta S = \Delta y \Delta x$ at a point in the liquid. There is a force acting on through this area element. The upper layer of fluid, that is moving faster, is trying to pull the lower layer with it. The lower layer, which is moving slower, is trying to slow down the upper layer. There is a viscous stress acting on the lower layer due to the upper layer. The relevant component is σ_{xz} , the x component of the force acting on a area-element that points along the z direction (you remember that a area-element is a vector whose direction is given by the outward pointing unit-normal). per-unit-area. Our everyday experience is codified in the following law

$$\sigma_{xz} = \eta \partial_z v_x = \eta \partial_z U \quad (14.1)$$

where η is the coefficient of viscosity which is positive. We shall revisit this point later.

Let us now try to understand this from a molecular picture. Before the details let me remind you of something you already know – how to understand the ideal gas laws from a molecular picture. To do this you want to calculate the pressure of a gas. The typical velocity of the gas molecules are set by their temperature, $mc^2 \sim k_B T$ where k_B is the Boltzman constant and T the temperature. In time Δt the number of molecule that hit an aread ΔS lies within a cylinder of volume



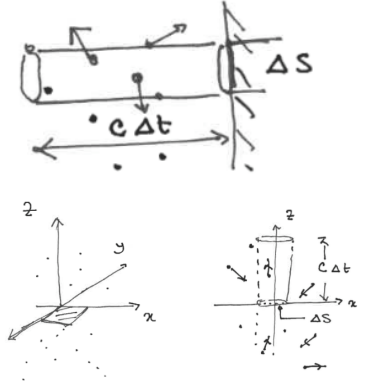
$c\Delta t\Delta S$. The number of molecules in this volume is $nc\Delta t\Delta S$ where n is the number density of the gas molecules. When the gas molecules hit the area ΔS they transfer to it a momentum equal to $2mc$ where c is their typical speed and m is their mass. Hence we get

$$\left(\begin{array}{c} \text{momentum} \\ \text{transferred} \\ \text{to area } \Delta S \\ \text{in time } \Delta t \end{array} \right) = 2mcnc\Delta t\Delta S \quad (14.2)$$

The pressure is then give by

$$p = \frac{1}{\Delta S\Delta t} \left(\begin{array}{c} \text{momentum} \\ \text{transferred} \\ \text{to area } \Delta S \\ \text{in time } \Delta t \end{array} \right) = 2mnc^2 \propto nk_B T \quad (14.3)$$

Now we apply a very similar argument to the area $\Delta S = \Delta x\Delta y$. This area has fluid on both sides. In time Δt a certain number of molecules pass through this area from the top to the bottom. A roughly equal amount of molecules go the other way round within the same amount of time. But the molecules coming from above carry higher x component of momentum (the upper layers are moving faster) than the lower ones, hence there is a net transfer of momentum from the upper layer to the lower one. This implies that the upper layers apply a force, more accurately a stress, to the lower one. Let me now put some math into words.



$$\begin{aligned} \left(\begin{array}{c} \text{x-momentum} \\ \text{transferred down-} \\ \text{ward through} \\ \text{area } \Delta S \text{ in time } \Delta t \\ \text{from a height } \pm \ell \end{array} \right) &= \left(\begin{array}{c} \text{No. of molecules} \\ \text{travelling from top} \\ \text{to bottom through} \\ \text{area } \Delta S \text{ in } \Delta t \end{array} \right) \left(\begin{array}{c} \text{x mome-} \\ \text{tum of each} \\ \text{molecule} \end{array} \right) - \left(\begin{array}{c} \text{No. of molecules} \\ \text{travelling from} \\ \text{bottom to top} \\ \text{through area } \Delta S \text{ in} \\ \Delta t \end{array} \right) \left(\begin{array}{c} \text{x momentum} \\ \text{of each} \\ \text{molecule} \end{array} \right) \\ &= (nc\Delta t\Delta S)(mc + mU(\ell)) - (nc\Delta t\Delta S)(mc + mU(-\ell)) \\ &= ncm[U(\ell) - U(-\ell)] \end{aligned}$$

We can estimate $U(\ell) - U(-\ell)$ by expanding $U(z)$ in a Taylor-series to obtain

$$U(\ell) - U(-\ell) = 2\frac{dU}{dz}\ell + \dots$$

What should we choose as ℓ in this equation? The molecules in a gas has a typical mean-free-path, which we call λ , the typical distances between collisions. If $\ell > \lambda$ then a molecule moving towards ΔS is likely to have a collision and diverted from actually reaching ΔS . So a reasonable approximation will be to replace ℓ by λ .

Then we obtain

$$\frac{1}{\Delta A}\Delta t \left(\begin{array}{c} \text{x-momentum} \\ \text{transferred down-} \\ \text{ward through} \\ \text{area } \Delta S \text{ in time } \Delta t \\ \text{from a height } \pm \ell \end{array} \right) = nmc\lambda \frac{dU}{dz} \quad (14.4)$$

The LHS is the stress – rate of transfer of momentum is x component of force, which per-unit-area is the stress – acting on the lower layer due to the upper one, hence

$$\sigma_{xz} = nmc\lambda \frac{dU}{dz}, \quad (14.5)$$

which gives the coefficient of viscosity to be

$$\eta = nmc\lambda = n\lambda\sqrt{k_B T} \quad (14.6)$$

Of course, there is nothing special about the x and z direction, if there is a shear – one component of velocity varies as a function of the other component – we expect viscous forces to try to smoothen the velocity differences.

14.1.1 Mean free time

Similar to the mean free path I can define a mean free time (τ), the time between two successive collision, such that $c = \lambda/\tau$. Now imagine a molecule with a cross-sectional area a^2 . In a time-window Δt it sweeps out a volume $ca^2\Delta t$. The number of molecules within this volume is $nca^2\Delta t$. These are the number of collisions in time Δt . If the time-window is the mean-free-time τ , then we expect a single collision, i.e.

$$nca^2\tau = 1 \quad (14.7)$$

The mean-free-path itself is then $\lambda = c\tau = 1/(na^2)$. Consequently,

$$\eta = \frac{1}{a^2}n\sqrt{k_B T} \quad (14.8)$$

14.1.2 Understanding viscosity

This is a wonderfully simple way to understand viscosity, a process that describes transport of one component of momentum along a direction orthogonal to it. To what extent can we trust the result ?

1. We have done a very rough calculation, we have not bothered to put even the right geometric factors. So the equality in Equation (14.8) is not really a proportionality. But it is not worth it to do this calculation in greater detail as is explained below.
2. It is almost impossible, given a molecule to get a precise number for a^2 , also we have assumed that the density of molecules are low and we have assumed that the molecules are those of an ideal gas. The collision cross section a^2 itself weakly depends on temperature.

3. There is actually a more precise way of calculating viscosity (and also thermal conductivity) for dilute gases systematically starting from what is known as the Boltzmann equation. This elementary course is not the place for such a calculation. I recommend a book [?] and a paper [Chen et al., 2001] for those of you who want to continue in this direction.
4. Most importantly, Equation (14.8) gives at least the right qualitative behavior of viscosity of gases as a function of temperature – viscosity increases with temperature – but not for liquids. Anyone who has ever cooked knows that the viscosity of liquids decreases with temperature.
5. The calculation of viscosity of a liquid or even a real gas from first principles – I tell you which molecules make up the fluid and you start calculating – is still practically impossible and is very much a topic of active research. Techniques similar to Chapman-Enskog expansion are used to understand the formation of planetary rings [Goldreich and Tremaine, 1982]. We use it to study formation of raindrops in warm clouds.
6. In modern fluid mechanics, a technique to simulate flows with complex boundaries is called the Lattice Boltzmann method. Here we do not solve the equations of the flow but numerically solve the Boltzmann equation itself on a grid. Then use the solution to directly calculate the velocity and density of the flow. This is a particularly easy and extremely efficient technique to solve for complex flows. Books and review articles on this topic are many, I recommend one by Thorne and Sukop ¹ that are accessible to beginners. These are often very useful source to learn the Chapman-Enskog expansion itself.
7. Recently, certain techniques derived (not rigorously) from string theory has been used to calculate the viscosity of strongly coupled (not dilute) systems, presumably from Quark-Gluon plasma. The results – which are sometime presented as signatures of great progress – are at the end not any better than what a simple dimensional estimate, like Equation (14.8) would give.

A historical note of particular importance to Swedish students. One of the first person to calculate the viscosity of a dilute gas was David Enskog. The calculation – which now goes by the name Chapman-Enskog expansion – was part of his PhD work at Uppsala.

¹MC Sukop and Jr. DT Thorne. Lattice boltzmann modeling. 2006

To summarise, if you want to have a qualitative picture of what viscosity is, the above calculation suffices. If you want to have a more detailed, quantitative calculation of viscosity of a material you are at the frontier of modern research.

14.1.3 Viscosity of liquids

As I commented above the viscosity of liquids follow a qualitatively different behavior than what we have obtained for dilute gases. Let us try to sketch out a plausible way to understand the viscosity of liquids. The key qualitative difference between liquids and gases is that in liquids the crystal structure of the solids, which melted to form liquid, is still somewhat preserved at least up to nearest neighbours. That means transport of a molecule is not as simple as that of gas. Before a liquid molecule can move it needs to break the (approximate) cage it typically finds itself in. This means it has to jump over a typical potential barrier. Which is possible because the molecule has thermal fluctuation. The mathematical problem of jumping over a potential barrier by a thermally driven particle is called the Kramer escape problem. Without going into details, we can say that the probability to jump over a barrier of energy U is the Boltzmann weight $\exp(-U/k_B T)$. Before it escapes it is approximately a particle trapped in a potential well of depth U and size a where a is the typical lattice spacing of the solid (that melted). Its frequency of oscillation in that potential is $\omega = \sqrt{2U/(ma^2)}$. Then we can expect the particle to jump over the cage in time $\tau \sim (1/\omega) \exp(U/k_B T)$. This then is the mean-free-time for liquids. The viscosity is then given by

$$\eta \sim nmc\lambda = nmc^2\tau = \frac{nmc^2}{\omega} \exp(U/k_B T) \quad (14.9)$$

This implies that the logarithm of viscosity is inversely proportional to temperature (in leading behavior). Given how hand-waving our arguments are it is surprising that $\eta = A \exp(B/k_B T)$ is actually a reasonable model for temperature dependence of viscosities in some liquids.

14.2 The Navier–Stokes equation

Having convinced ourselves that viscous phenomenon exists and are important how do we include them in the Euler equation and reach a more complete theory of fluids? This we proceed to do now. In the last chapter we already wrote the Euler equation in a form that is suitable for this generalisation

$$\partial_t(\rho v_\alpha) + \partial_\beta [\rho v_\alpha v_\beta - \sigma_{\alpha\beta}] = 0 \quad (14.10)$$

So far, within $\sigma_{\alpha\beta}$ we have considered only the hydrostatic pressure. Now we have a new viscous contribution to this stress, given by

$$\sigma_{\alpha\beta} = (\text{viscosity}) S_{\alpha\beta}. \quad (14.11)$$

Let us see how we arrive to this expression. Obviously the generalisation of Equation (14.1) is

$$\sigma_{\alpha\beta} = (\text{viscosity}) \frac{\partial v_{\alpha}}{\partial \beta} \quad (14.12)$$

In the first chapter we defined the strain tensor from a general deformation tensor. In fluid dynamics, as we have mentioned before, it is the rate-of-strain not the strain itself that matters. The strain tensor was given by

$$s_{\alpha\beta} = \frac{1}{2} [\partial_{\beta} u_{\alpha} + \partial_{\alpha} u_{\beta}] . \quad (14.13)$$

Replacing $\mathbf{u}$ by the rate-of-deformation (more commonly known as velocity) $\mathbf{v}\Delta t$, we get the rate-of-strain tensor

$$S_{\alpha\beta} = \frac{1}{2} [\partial_{\beta} v_{\alpha} + \partial_{\alpha} v_{\beta}] . \quad (14.14)$$

The RHS of Equation (14.12) must contain the symmetric $S_{\alpha\beta}$ and not the more general $G_{\alpha\beta} \equiv \partial_{\beta} v_{\alpha}$ because the latter can always be decomposed into a symmetric and an anti-symmetric part. The symmetric part is just $S_{\alpha\beta}$ the anti-symmetric part is proportional to vorticity, $\boldsymbol{\omega}$. Remember that the anti-symmetric part denotes a rotation, which is *not* a deformation but a rigid rotation. The stress cannot depend on rigid rotation and rigid rotation implies that there is no relative motion between different elements of the same body hence no stress develops. So we conclude that the viscous stresses are related to the rate-of-strain tensor by

$$\sigma_{\alpha\beta} = \text{viscosity} S_{\mu\nu} \quad (14.15)$$

The *viscosity* is then, in principle a fourth-rank tensor $\eta_{\alpha\beta\mu\nu}$ not just one number as in Equation (14.8) but $3^4 = 81$ numbers ! If you look at any handbook of fluid properties, at most two viscosities are given. How do we square that circle ? This is done by the magic of symmetry.

14.2.1 Isotropic fluids

Fluid have no internal lattice structure – a definition of being a fluid – hence whatever property we calculate must remain invariant under space rotation. A particular fluid molecule itself may not be spherically symmetric, but can have quite a wierd shape. But there are many of those in many different orientation in a fluid parcel. Hence we expect even for non-isotropic molecules the bulk properties of a fluid must be invariant under rotation.

The viscosity, $\eta_{\alpha\beta\mu\nu}$ must be a fourth-order but *isotropic tensor*. Let us try to understand that statement first in a simple context.

In solids, a generalization of Hook's law can be written as

$$\sigma_{\alpha\beta} = C_{\alpha\beta\mu\nu} s_{\mu\nu} \quad (14.16)$$

where $\sigma_{\alpha\beta}$ is the stress and $s_{\mu\nu}$ is the strain (not the rate-of-strain) tensor. The fourth-rank tensor that sits between them can be quite complex. For solid with no particular symmetry it can have 21 independent numbers. For isotropic solids again we land up with two numbers which are called the two Lamé coefficients.

What is a isotropic second rank tensor ? The tensor quadratic of such a tensor must remain invariant under rotation, hence it must be a sphere. Hence any isotropic second rank tensor must be proportional to the simplest isotropic second rank tensor $\delta_{\alpha\beta}$. So although a general second-rank tensor has 9 number as isotropic one has just one. How do we construct a fourth rank isotropic tensor ? The simplest is of course to write it as products of isotropic second rank ones

$$\eta_{\alpha\beta\mu\nu} = A\delta_{\alpha\beta}\delta_{\mu\nu} + B\delta_{\alpha\mu}\delta_{\beta\nu} + C\delta_{\alpha\nu}\delta_{\beta\mu} \quad (14.17)$$

Thus we have landed up with three independent number A,B and C instead of 81. There is still one more trick necessary to reduce it to two. That is to assume that space reflection $x \rightarrow -x$, $y \rightarrow -y$ and $z \rightarrow -z$ should also keep the viscosity invariant – this will yield $B = C$. So we end up with two independent numbers. The usual convention is to write them in the following manner

$$\eta_{\alpha\beta} = \mu(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\delta_{\alpha\beta}\partial_\gamma v_\gamma + \zeta\delta_{\alpha\beta}\partial_\gamma v_\gamma) \quad (14.18)$$

with two independent coefficients of viscosity, μ which is called the coefficient of shear viscosity and ζ called the coefficient of bulk viscosity. As we shall mostly consider incompressible flows the viscous stress tensor for us will reduce to

$$\sigma_{\alpha\beta} = \mu [\partial_\beta v_\alpha + \partial_\alpha v_\beta] \quad (14.19)$$

substituting this back in Equation (14.10) we obtain

$$\partial_t(\rho v_\alpha) + \partial_\beta [\rho v_\alpha v_\beta + p\delta_{\alpha\beta} - \mu(\partial_\alpha v_\beta + \partial_\beta v_\alpha)] = 0 \quad (14.20)$$

On simplification this equation reduced to

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{v}, \quad (14.21)$$

which is the celebrated Navier–Stokes equation. Here we have replaced μ/ρ by ν . The viscosity μ is called the dynamic viscosity and ν the kinematic viscosity.

14.2.2 Reynolds number and similarity

There is a wonderful property of the Navier–Stokes equation that I shall point out now. But before proceeding it is useful to take a curl of the equation and write it in vorticity formulation:

$$\partial_t \boldsymbol{\omega} + \nabla \times (\boldsymbol{\omega} \times \mathbf{v}) = \nu \nabla^2 \boldsymbol{\omega} \quad (14.22)$$

Now think of a specific flow. For example a flow about a sphere. For example a metal sphere sedimenting in silicone oil. Typically, after

a little while the sphere attains a constant velocity, called terminal velocity. If the walls of the container, where this experiment is being made, are far away from the sphere, then there are only three physical quantities that set the problem: (a) the diameter of the sphere – could also be its radius, it does not matter, (b) its terminal velocity and (c) the viscosity of the oil. Let us use the diameter D as the unit of length, the terminal velocity V as the unit of velocity. Then the unit of time becomes $T = D/V$. Let us now introduce non-dimensional quantities

$$x = \tilde{x}D, \quad y = \tilde{y}D, \quad z = \tilde{z}D, \quad t = \tilde{t}T \quad (14.23)$$

This implies that

$$\frac{\partial}{\partial x} = \frac{\partial \tilde{x}}{\partial x} \frac{\partial}{\partial \tilde{x}} = \frac{1}{D} \frac{\partial}{\partial \tilde{x}} \quad (14.24)$$

and

$$\frac{\partial}{\partial t} = \frac{1}{T} \frac{\partial}{\partial \tilde{t}} \quad (14.25)$$

The velocity and vorticity in these new units are measured as

$$\boldsymbol{\omega} = \frac{1}{T} \tilde{\boldsymbol{\omega}}, \quad \mathbf{v} = V \tilde{\mathbf{v}} \quad (14.26)$$

Substituting equations 14.23 to 14.26 to (14.22) we obtain

$$\begin{aligned} \frac{1}{T^2} \tilde{\partial}_t \tilde{\boldsymbol{\omega}} + \frac{V}{DT} \tilde{\nabla} \times (\tilde{\boldsymbol{\omega}} \times \tilde{\mathbf{v}}) &= \nu \frac{1}{TD^2} \tilde{\nabla}^2 \tilde{\boldsymbol{\omega}} \\ \tilde{\partial}_t \tilde{\boldsymbol{\omega}} + \tilde{\nabla} \times (\tilde{\boldsymbol{\omega}} \times \tilde{\mathbf{v}}) &= \frac{1}{\text{Re}} \tilde{\nabla}^2 \tilde{\boldsymbol{\omega}} \end{aligned} \quad (14.27)$$

where $\text{Re} \equiv DV/\nu$ is a dimensionless number called the Reynolds number. This shows that the flow is not a function of the diameter, the velocity and the viscosity alone but an unique, dimensionless combination of them. There is only one dimensionless parameter that the Navier–Stokes equations really depends on, that is the Reynolds number. So if you have two situation, where the characteristic length scale, the velocity scale and the viscosity of the liquid changes but the Reynolds number remains the same you shall have exactly the same answer in terms of *dimensionless* velocity and vorticity. In fluid dynamics this is called the similarity principle.

14.2.3 Non-Newtonian liquids

There is nothing typical about our assumption that the stress and the rate-of-strain are proportional to each other. The coefficient of viscosity itself could be a function of the rate-of-strain. As an example, consider jam. When you try to pour jam, a little stress may not produce any motion at all implying that the viscosity is very high. But a little higher stress, called *yield stress* makes it flow. So the viscosity actually decreases with imposed stress for small

The non-dimensionalization described here is one of the first steps a physicist takes while writing down a mathematical description of natural phenomenon. Otherwise you have no idea of what is big and what is small. What you can ignore in the first approximation and what forms the core of your problem. The minimum you should learn from any physics course is how to “think like a physicist” and looking for characteristic scales of a natural phenomenon is the first step in that. The usual SI units are one where we are roughly two in units of length, about hundred in unit of mass and our characteristic time (set by our pulse) is about unity. So these are appropriate if you want to know how long something in nature is compared to how tall you are, or how heavy something is compared to how heavy you are. But if you want to think about the Sun or an atom, your length scales are totally inappropriate. You have to find the *natural* length and time scales to even start thinking about the problem. The CGS units could be called the characteristic units of a cockroach.

stress. Such a fluid is called a *shear-thinning* fluid. Now consider a corn-starch solution. If you move very fast you may be able to walk on corn-starch, but if you stand still you shall sink in a bathtub full of corn-starch solution. In such a material when the rate-of-strain increases the viscosity also increases – these are *shear-thickening* material. For an introduction see the Physics Today article by Wagner and Brady [Wagner and Brady, 2009]. We shall revisit the problem of effective viscosity of a suspension later in these lectures.

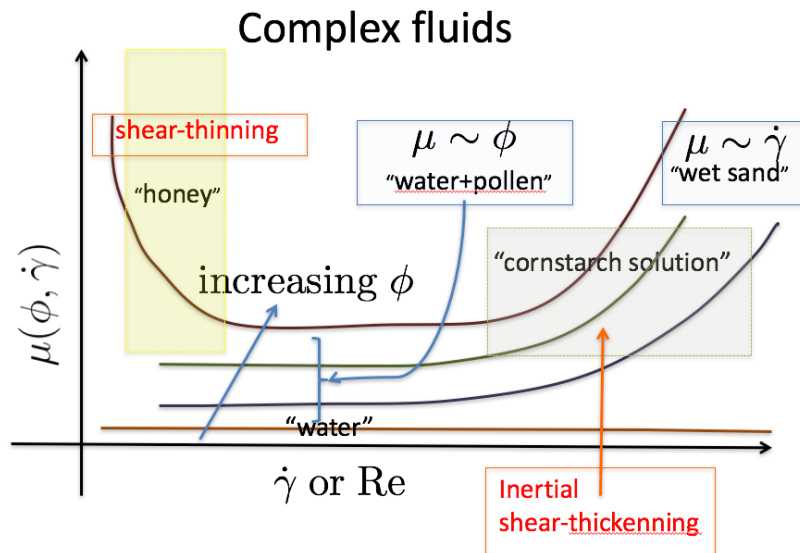


Figure 14.1: A cartoon of viscosity versus rate-of-strain diagram for different families of suspension. The vertical axis is the effective viscosity, μ , the horizontal axis is the rate-of-strain, $\dot{\gamma}$, ϕ is the volume-fraction occupied by whatever is being added to water to make this suspension. This sketch is simplistic, merely for illustration although qualitatively correct.

14.3 Boundary conditions

The Navier–Stokes equation is a second order equation in space, hence it needs two boundary conditions ². In what follows, let us consider only rigid boundaries, not free surfaces or surfaces in contact with soft boundaries. Obviously the component of velocity normal to a solid boundary must be zero. This is the boundary condition used in Euler equation too. What about the component perpendicular to the boundary? It is reasonable to set that to zero too – the no-slip boundary condition. There are cases where a slip can appear, but that is not something we shall deal with in this course.

²Euler was a first order equation in space, hence needed only one boundary condition. Does that make you suspicious?

Exercise 5

1. Viscosity of liquids

Consider the following liquids, air, water, mercury, olive oil and glycerine. From a handbook find out their dynamic shear viscosity μ and density at room temperature. We consider

incompressible flows of each of these liquids in the same geometry, order them from the most viscous to the least viscous one.

2. Energy dissipation rate

Consider an incompressible (density $\rho = 1$) fluid with no external forces, with an initial velocity $\mathbf{v}(\mathbf{r})$ as a function of space coordinate $\mathbf{r}$. The flow is such that the velocity goes to zero at infinity. Show that the rate of energy dissipation, defined by

$$\varepsilon = \partial_t \int \frac{1}{2} v^2 dV = -\nu \int |\boldsymbol{\omega}|^2 dV \quad (14.28)$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ and the integral is over all space. In other words, the nonlinear term in Navier–Stokes conserves energy. You already know this from Bernoulli’s equation.

3. Symmetries of the stress tensor

You may have already noticed something amiss in my description of the stress and (rate-of-strain). The rate-of-strain is clearly a symmetric tensor whereas so far I have presented no evidence that stress tensor is a symmetric one. This can be proved too. That is the task of this exercise. There are several way to prove this, some are intuitive and geometric, but here I take a more algebraic approach as a practice of your tensorial tools. The crucial physical input is the conservation of angular momentum. If angular momentum is a conserved quantity they we should be able to write a conservation equation of it in the form

$$\partial_t L_\alpha + \partial_\beta K_{\alpha\beta} = 0 \quad (14.29)$$

where

$$\mathbf{L} = \rho \mathbf{r} \times \mathbf{v} \quad (14.30)$$

is the density of angular momentum where $\mathbf{r} = (x_1, x_2, x_3)$ is the position vector. The trick is to write an equation for $\mathbf{L}$ and try to cast it in the form in Equation (14.29). To do this we shall use the conservation of linear momentum which we write in the form

$$\partial_t (\rho v_\alpha) + \partial_\beta [\rho v_\alpha v_\beta - \sigma_{\alpha\beta}] = 0. \quad (14.31)$$

Here the stress tensor $\sigma_{\alpha\beta}$ contains the pressure as its diagonal element. Our aim is to demonstrate the symmetry of this stress tensor as a consequence of the conservation of the angular momentum.

Calculate $\partial_t L_\alpha$. After some algebra (show the steps) you shall land up with the following expression

$$\partial_t L_\alpha + \partial_\mu (v_\mu L_\alpha) + \partial_\mu (\epsilon_{\alpha\beta\gamma} x_\beta \sigma_{\mu\gamma}) = \epsilon_{\alpha\beta\gamma} \sigma_{\beta\gamma} \quad (14.32)$$

The left hand side of this equation looks like a conservation law. There are two contributions to the flux – the first one is due to advection by the flow, the second one you can interpret the divergence of the “torque of the stress tensor”. But whatever is on the RHS is not in the form of divergence of any quantity. So it must be zero. The only way it can be zero in general is if $\sigma_{\beta\gamma}$ is symmetric.

To get to this result you may need to juggle a bit the spatial and time derivatives. Remember that $\mathbf{r}$ and t are independent coordinates as you are considering Eulerian fields and not Lagrangian particles; also remember that $\frac{\partial x_\mu}{\partial x_\beta} = \delta_{\mu\beta}$

15 Steady Viscous flows

15.1 Steady unidirectional flows

The central feature of Equation (14.21), that you must have noticed already is that it is non-linear. The mathematical difficulties due to this nonlinear nature are so severe that in most cases we resort to a situation where for one reason or the other the equations become linear. One such trick is to impose symmetries on the problem such that the nonlinear term effectively disappears.

15.1.1 Poiseuille flow

The simplest example of these are steady flows that depend on only one coordinate direction. For example consider a flow in a pipe with circular cross-section. There is a pressure gradient applied across the pipe. The pipe is quite long and we are staying away from either of its two ends. Let us use cylindrical coordinates with the axis of the coordinate coinciding with the axis of the pipe. Let the radius of the pipe be a . The velocity of the flow must go to zero at $r = a$, hence we have the boundary condition that

$$v_r = v_\theta = v_z = 0 \quad \text{at} \quad r = a \quad (15.1)$$

The pressure is specified at the two ends of the pipe. Let us now write down the Navier-Stokes equation for the z component of velocity

$$v_r \partial_r v_z + \frac{v_\theta}{r} \partial_\theta v_z + v_z \partial_z v_z = -\partial_z p + \mu \left(\partial_r^2 v_z + \frac{1}{r^2} \partial_\theta^2 v_z + \partial_z v_z + \frac{1}{r} \partial_r v_z \right) \quad (15.2)$$

where we have assumed that the flow is steady. From the symmetry of the problem, we expect that the only component of the flow is v_z , v_r and v_θ are zero and v_z is a function of r alone. With this last assumption (15.2) simplifies to

$$0 = -\partial_z p + \mu \left(\partial_r^2 + \frac{1}{r} \partial_r \right) v_z \quad (15.3)$$

As the flow is symmetric under translation along the axis of the pipe, $\partial_z p$ must be a constant, which I write as

$$\partial_z p = \frac{p_{\text{in}} - p_{\text{out}}}{L} \equiv G \quad (15.4)$$

where L is the length of the pipe and the two pressures are the pressures at its inlet and outlet. Substituting this in (15.3) we get

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}\right) v_z(r) = -\frac{G}{\mu} \quad (15.5)$$

This can be integrated to obtain

$$v_r = \frac{G}{4\mu} \left(-r^2 + A \log r + B\right) \quad (15.6)$$

With the boundary condition, $v_z(a) = 0$. If $A \neq 0$ then the solution is singular at $r = 0$, which cannot be the case, hence we choose $A = 0$. The boundary condition at a sets B such that we now have

$$v_z(r) = \frac{G}{4\mu} (a^2 - r^2), \quad (15.7)$$

a nice parabolic profile. So we have finally solved one flow problem. This one, called Poiseuille flow is crucially important. Using v_z we can now calculate the flux material through the pipe to be

$$Q = \int_0^a v_z(r) 2\pi r dr = \frac{\pi a^4 (p_{\text{in}} - p_{\text{out}})}{8\mu L} \quad (15.8)$$

This expression has been used to measure viscosity of liquids for quite a long time.

15.1.2 Taylor–Couette flow

Let us start by solving another canonical problem in fluid mechanics. The flow between two rotating cylinders. We assume the cylinders to be infinitely long, hence we ignore the edge effects from finite cylinders. The cylinders have radii a (inner one) and b (outer one) rotating with angular velocities Ω_a and Ω_b . We look for steady flow solutions. We assume that the symmetries of the boundary condition are reflected in the solution being fully aware that such an assumption is valid only at small Reynolds number. The symmetry dictates that there is no flow along the axis of the cylinder. There is no flow along the radial direction either. The only component of the flow is along the azimuthal direction. Continuing in a manner similar to the case of pipe flow in the previous chapter we obtain the following result.

The only non-zero component of velocity is $v_\phi = \omega r$ where $\omega(r)$ is itself a function of the r coordinate (distance from the axis) alone

$$\omega = -\frac{A}{2r^2} + B. \quad (15.9a)$$

$$A = \frac{2a^2b^2}{b^2 - a^2} (\Omega_b - \Omega_a) \quad (15.9b)$$

$$B = \frac{b^2\Omega_b - a^2\Omega_a}{b^2 - a^2} \quad (15.9c)$$

The details of the calculation are left as an exercise.

15.1.3 Where I cheated

The calculation above looks straightforward enough, then you may ask why we make such a song and dance about solving the Navier–Stokes equation, in some cases – like the one above – it looks simple enough. What looks simple, is actually quite wrong. I have cheated above by demanding that the symmetry of the boundary condition must be reflected in the symmetry of the solution. This is true for linear equations but not at all for nonlinear systems. Also there is no longer any uniqueness theorem for solutions. We have found just one solution, nothing tells us that there are no other solutions or even if there are which solution is the one that nature chooses? The fact of the matter is that the solution above works only if we assume that the nonlinear term is not important. The fact that the symmetry dictates the nonlinear term to be zero, in principle, mean nothing. As is also seen in practice, if you open a tap you do not see the nice flow profile as found in (15.7). To see the parabolic you have to look at a situation where the nonlinear term is not important, in other words for highly viscous flows. That does not mean that you need a highly viscous liquid you just need the Reynolds number to be very small. You can do that with water by taking a very very narrow channel. This is the domain of microfluidics.

Exercise 6

1. **Taylor-Couette flow** The flow between two rotating cylinder, as described in section 15.1.2 is named after Geoffrey Ingram Taylor and Maurice Marie Alfred Couette. We shall need this flow later in our discussions in instabilities. Your job in this exercise is to find out the solution given in section 15.1.2. By the symmetry of the problem, show that the contribution from the non-linear term of the Navier–Stokes equation is zero. Then look for steady solutions. Use cylindrical coordinates with the axis running along the common axis of the two cylinders. Argue from symmetry that the only component of velocity is the

azimuthal component $v_\phi = \omega r$ where ω is a function of r only. The Navier–Stokes equation will now simplify to an ordinary differential equation in the radial coordinate r which you can easily solve. Then set the boundary condition at $r = a$ and b . Once you have found the solution consider the following case. The inner cylinder is rotating with an angular velocity Ω and the outer cylinder is being held fixed. Calculate the torque you need to apply to the outer cylinder ?

16 Creeping Flows

16.1 The Stokes equations

In the last lecture, we solved one of the simplest and most common flow problem : the Poiseuille flow. This falls within a class of flows that are steady and unidirectional. This symmetry dictated that the nonlinear term is zero and the problem was reduced to one where we needed to solve the following set of equations

$$\mu \nabla^2 \mathbf{v} = \nabla p \quad (16.1a)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (16.1b)$$

I also told you that the symmetry comes from the boundary conditions and there is no reason why the the solution should have the same symmetry as the boundary conditions – the Navier–Stokes is a fundamentally non-linear problem. And indeed the solution we found although useful is not what is often seen in pipe flows, typical pipe flows are turbulent.

But there are many cases where (16.1) turns out to be a very useful approximation. This is not due to symmetry but because the Reynolds number of the problem is very very small. It is clear that when the Reynolds number is small we can ignore the nonlinear term in the Navier–Stokes equation – to see this let us make rough estimates of the nonlinear term and the viscous term,

$$\mathbf{v} \cdot \nabla \mathbf{v} \sim U \frac{1}{L} U \sim \frac{U^2}{L} \quad , \quad (16.2)$$

$$\nu \nabla^2 \mathbf{v} \sim \nu \frac{1}{L^2} U \quad , \quad (16.3)$$

whose typical speed is U , typical length scales are L and ν is the kinematic viscosity. The ratio of these two term is

$$\frac{\mathbf{v} \cdot \nabla \mathbf{v}}{\nu \nabla^2 \mathbf{v}} \sim \frac{U^2 L^2}{\nu U L} \sim \frac{U L}{\nu} = \text{Re}, \quad (16.4)$$

the Reynolds number. This is a second interpretation of the Reynolds number. Batchelor mentions [Batchelor, 1967] that it is actually Stokes

who first introduced the number that bears Reynolds's name although Reynolds is indeed the person who pointed out its usefulness in study transition of flows from laminar to turbulent. Next, let us estimate the strength of the time derivative

$$\partial_t \mathbf{v} \sim \frac{U}{T} \sim \frac{U^2}{L} \quad (16.5)$$

where we have assumed that the characteristic time scale T is set by the velocity of the flow and the characteristic length. This may not always be the case. For example you put your spoon in a cup and stir it the time derivative is determined by how fast you stir the spoon. At the moment let us ignore those cases where the time scale is set by an external agent, e.g., you and consider only those cases where the characteristic time scale is set by the flow itself. Then the estimate at (16.5) remains valid. Then the ratio between the time-derivative and the viscous term is again give by

$$\frac{\partial_t \mathbf{v}}{\nu \nabla^2 \mathbf{v}} \sim \text{Re}, \quad (16.6)$$

the dimensionless Reynolds number. The set of equations, which emerges from Navier–Stokes once we ignore the time derivative and the nonlinear term is called the Stokes equation.

When is the Stokes equation the correct approximation of a flow problem ? The standard answer is, when the Reynolds number is very small. How small do we mean when we say very small ? That is a tricky question. The Stokes flow is a good approximation when the Reynolds number is zero, it may be a good approximation for even non-zero Reynolds numbers but when it stops being a good approximation depends on which flow we are talking about. In all cases that I know of, it is safe to use the Stokes equation when the Reynolds number is less than unity. As the Reynolds number increases, the steady Stokes flow stops becoming valid and a kaleidoscope of phenomenon start appearing. The study of these phenomenon forms a large part of fluid mechanics, they fall under the general class of *transition to turbulence* problems. We shall spend quite some time with them, but for now we shall study the Stokes problem in great detail.

Why should we do that ? Well, one answer is that this is the realm where the equations of a flow are linear, so all our mathematical machinery works. The second answer is that often solutions to the Stokes equation are necessary to understand what happens as the Reynolds number increases. But the third and the most important answer is that this realm is of practical importance for at least three kinds of problems,

- Swimming microorganisms: A sperm or a bacterial swims; the medium is essentially water, $\nu = 10^{-2} \text{cm}^2/\text{s}$, typical length scales $L = 1 \mu\text{m}$ and typical speeds $U = 30 \mu\text{m}/\text{s}$ which gives a Reynolds number of $\text{Re} \approx 10^{-4}$. A fantastic introduction to this problem is the famous article by E.M. Purcell called “Life at low-Reynolds number” [Purcell, 1977].
- Microfluidics: Microfluidic is a subject where fluid – typically water – flows through channels of micro-meter width, see e.g., a review article [Squires and Quake, 2005] by Squire and Quake for an introduction. Fluids carrying different chemicals can move in a device and meet different chemicals on their way, thus a whole sequence of chemical reactions are run on a small chip the size of a microscope slide. Water is the typical working fluid – $\nu = 10^{-2} \text{cm}^2/\text{s}$ at room temperature and atmospheric pressure – typical velocities ranges from $1 \mu\text{m}/\text{s}$ to $1 \text{cm}/\text{s}$ and typical lengths are from $1 - 100 \mu\text{m}$. The typical Reynolds number ranges from 10^{-6} to about 10.
- Flow through porous rocks: This could be seepage of water through very thin channels which form a complicated network. It could also be flow of water through a sponge, in which case depending on the stresses generated by the fluid the sponge itself can deform, channels can widen or shrink. The typical Reynolds number is again very very small compared to unity.

16.2 General remarks

Before I get into the details of solving the Stokes equations in different situations, let me make some general remarks.

16.2.1 What is inertia-less flow?

As you have realized the Reynolds number is an estimate of the inertial term to the viscous term. In these flows the inertia is very very small. What does that actually mean? It means that you can essentially think about Aristotelean mechanics in this regime. The force is essentially balanced by velocity. In our realm, if I stop paddling my bike it takes some distance (and some time) to get to a stop. This distance depends on my inertia and also the friction between the tires of the bike and the road. In the Stokes realm the as soon as I stop paddling, I would come to a stop very quickly due to the viscous forces.

16.2.2 *Linearity*

The Stokes equations are linear. Which implies that the mathematical machinery needed to understand the content of these equations are well known. You may have already learned some of those in courses in Electrodynamics or mathematical physics.

One, somewhat obvious, consequence is that the solution depends linearly on the boundary values of velocity or pressure. For example, think about a sphere moving in a liquid with uniform velocity U . The solution everywhere should depend linearly on U , it cannot depend on any higher powers of U .

The linearity has few far-reaching consequences, which we shall describe in turn.

16.2.3 *Uniqueness*

One of which is the *uniqueness*. Given a set of boundary conditions, there is only one solution. Hence if you can somehow guess a solution that fits the boundary conditions you have found the one and only solution. The details of the proof is given in the appendix to this chapter.

16.2.4 *Superposition*

In electrostatics you have come across the superposition principle, if you want the electric field (or potential) at one point in space due to a distribution of charges, this field is the same as the sum of the fields from individual charges. The same idea works here too. Consider a sphere that is moving with a constant velocity, $V\hat{z}$. Assume that I have given you the flow everywhere, both pressure and velocity, due to this translating sphere. Now consider a problem where the same sphere is not translating but rotating about an axis. I have told you the solution in that case too. Now to find the solution for a sphere that is both rotation and translating you just need to add these two solutions. A note of caution though, if you want to make a new solution as a superposition of two solutions you must be careful that the two solutions you add up satisfy the same or at least compatible boundary conditions.

16.A *Proof of uniqueness of solutions of Stokes equation*

Let me prove this, to show you how such proof generally go. Consider a domain V with boundary Ω . The boundary can be at infinity. Let us assume that within this domain we have two solutions of the Stokes problem, a velocity-pressure pair u, ϕ and v, p and two

corresponding rate-of-strain tensors

$$E_{\alpha\beta} = \frac{1}{2} (\partial_\beta u_\alpha + \partial_\alpha u_\beta) \quad \text{and} \quad S_{\alpha\beta} = \frac{1}{2} (\partial_\beta v_\alpha + \partial_\alpha v_\beta). \quad (16.7)$$

Both of them satisfy the same boundary condition such that on Ω

$$\mathbf{u} = \mathbf{U}^* \quad \text{and} \quad \mathbf{v} = \mathbf{U}^* \quad (16.8)$$

$$\phi = p^* \quad \text{and} \quad p = p^* \quad (16.9)$$

Now consider the volume integral of the square of the strain tensor

$$\begin{aligned} & \int_V dV [E_{\alpha\beta} - S_{\alpha\beta}] [E_{\alpha\beta} - S_{\alpha\beta}] \\ &= \int_V dV \frac{1}{4} [(\partial_\beta u_\alpha - \partial_\beta v_\alpha) + (\partial_\alpha u_\beta - \partial_\alpha v_\beta)] [\dots] \\ &= \int_V dV \frac{1}{4} \{ \partial_\beta (u_\alpha - v_\alpha) [\dots] + (\dots) [\dots] \} \\ &= -\frac{1}{4} \oint_\Omega dS n_\alpha (u_\alpha - v_\alpha) [\dots] - \int_V dV \frac{1}{4} [u_\alpha - v_\alpha] \partial_\beta [\dots] - \frac{1}{4} \oint_\Omega dS n_\beta (u_\beta - v_\beta) [\dots] - \int_V dV \frac{1}{4} [u_\beta - v_\beta] \partial_\alpha [\dots] \end{aligned} \quad (16.10)$$

The final step follows from integration by parts applied to three dimensional integral or in other words application of Gauss's divergence theorem. The two surface integrals are zero by virtue of (16.9) because on Ω both $\mathbf{v}$ and $\mathbf{u}$ are equal to $\mathbf{U}^*$, hence the integrand is everywhere zero on the surface. Continuing our calculation

$$\begin{aligned} & \int_V dV [E_{\alpha\beta} - S_{\alpha\beta}] [E_{\alpha\beta} - S_{\alpha\beta}] \\ &= -\frac{1}{4} \int_V dV [u_\alpha - v_\alpha] \partial_\beta [\dots] - \frac{1}{4} \int_V dV [u_\beta - v_\beta] \partial_\alpha [\dots] \\ &= -\frac{1}{4} \int_V dV [\dots] \left[\partial_\beta^2 (u_\alpha - v_\alpha) + \partial_\beta \partial_\alpha (u_\beta - v_\beta) \right] - \frac{1}{4} \int_V dV [\dots] \partial_\alpha [\dots] \end{aligned} \quad (16.11)$$

The term in red is zero because both $\mathbf{u}$ and $\mathbf{v}$ are incompressible. Furthermore as both $\mathbf{u}$ and $\mathbf{v}$ are solutions of the Stokes equations,

$$\partial_\beta^2 u_\alpha = \frac{1}{\mu} \partial_\alpha \phi \quad \text{and} \quad \partial_\beta^2 v_\alpha = \frac{1}{\mu} \partial_\alpha p \quad (16.12)$$

Substituting this back into (16.11) we obtain

$$\begin{aligned} & \int_V dV [E_{\alpha\beta} - S_{\alpha\beta}] [E_{\alpha\beta} - S_{\alpha\beta}] \\ &= -\frac{1}{4} \int_V dV [u_\alpha - v_\alpha] \left[\frac{1}{\mu} \partial_\alpha (\phi - p) \right] - \frac{1}{4} \int_V dV [\dots] \partial_\alpha [\dots] \\ &= -\frac{1}{4\mu} \oint_\Omega dS (\phi - p) (u_\alpha - v_\alpha) + \frac{1}{4\mu} \int_V dV (\phi - p) \partial_\alpha (u_\alpha - v_\alpha) - \frac{1}{4} \int_V dV [\dots] \partial_\alpha [\dots] \end{aligned} \quad (16.13)$$

Again we have used integration by parts, the surface integral is zero by virtue of identical boundary condition for the two solutions and the term in red is zero because of incompressibility. The fate of the

last volume integral is also the same by exactly same treatment. So finally we obtain

$$\int_V dV [E_{\alpha\beta} - S_{\alpha\beta}] [E_{\alpha\beta} - S_{\alpha\beta}] = 0 \quad (16.14)$$

The integrand in the integral on the LHS of this equation is always non-negative. Hence the only way this relation can be satisfied is if $E_{\alpha\beta} = S_{\alpha\beta}$ everywhere within the volume V . So if the Stokes solution has two solutions which satisfy the same boundary condition the rate-of-strain produced by the two solutions are identical everywhere. So the only way these solutions can differ from each other is either by rigid rotation or translation both of which are ruled out by boundary conditions. Hence the two solutions must be the same everywhere. This proves the uniqueness theorem. The uniqueness theorem for other linear problems, e.g., the Laplace's equation is also proved in a similar manner. You only need to learn this trick once.

17 *Creeping Flows in two dimensions*

17.1 *Possible simplifications*

The key idea is to try and see if we can reduce the mathematical complexity of the problem.

17.1.1 *Equation for vorticity*

Let us take a curl of (16.1a) to obtain

$$\nabla^2 \omega = 0 \quad (17.1)$$

which is the vector Laplace's equation. This is a simpler problem than (16.1) as you have reduced the number of variables by one. But now you have to figure out the boundary condition in terms of ω and also invert the definition of vorticity

$$\omega = \nabla \times \mathbf{v} \quad (17.2)$$

to obtain the velocity field. So it is not clear whether we have gained anything.

17.1.2 *Equation for pressure*

Take a divergence of (16.1a) to obtain

$$\nabla^2 p = 0 \quad (17.3)$$

which is the usual Laplace's equation. In electrostatics, the potential satisfies the same equation. Thus, if you know the pressure boundary conditions, you now have a straightforward way of finding the pressure everywhere. In reality, you often know the velocity boundary condition and not the pressure one.

17.1.3 *Vector Streamfunction*

For example, let us try to choose the velocity in such a way that the incompressibility constraint is satisfied automatically. One way this

can be done is by choose the velocity to be the curl of another vector field, that I call Ψ , such that

$$\mathbf{v} = \nabla \times \Psi. \quad (17.4)$$

Taking another curl we obtain,

$$\boldsymbol{\omega} = \nabla \times \nabla \times \Psi = -\nabla^2 \Psi \quad (17.5)$$

if we assume $\nabla \cdot \Psi = 0$, which is the equivalent of choosing a gauge in magnetostatics. Substituting (17.5) in (17.1) I get

$$\nabla^2 \nabla^2 \Psi = 0 \quad (17.6)$$

This is a single equation for again a vector function. The advantage over the formulation in Equation (17.1) is that once you know the streamfunction you can find the velocities by calculating derivatives, not integrals. Also the velocity boundary conditions implies conditions on the first derivative of the streamfunction. The disadvantage is that you have to now solve for a fourth order partial differential equation. It will turn out that if by the symmetry of the problem we can ignore one dimension among the three, we can formulate the problem in terms of a *scalar* streamfunction. This would provide significant simplification to the problem as I show in the next section.

17.2 In two dimensions

Linearity also means that the symmetry of the boundary conditions are reflected in the solution. This is a particularly powerful way to solve many problems that arise in Stokes flows. The flow is independent of at least one coordinate direction, being essentially limited a plane – two dimensions. Clearly the vorticity is now a scalar as it points either up or down. To keep things simple, let us consider it to be the $x - y$ plane with z axis pointing vertically upward. As nothing is a function of the z direction, we start by defining a streamfunction such that

$$\Psi = \hat{z}\psi(x, y) \quad (17.7)$$

where ψ is a scalar function of x, y . The velocity is obtained by the curl of this streamfunction, such that,

$$v_x = \partial_y \psi \quad \text{and} \quad v_y = -\partial_x \psi \quad (17.8)$$

The vorticity is given by

$$\boldsymbol{\omega} = \hat{z}\omega(x, y) = -\hat{z}(\partial_x^2 \psi + \partial_y^2 \psi). \quad (17.9)$$

The equation for the scalar streamfunction, ψ is then given by

$$\Delta_{\perp}^2 \psi = 0 \quad (17.10)$$

where $\Delta_{\perp}$ is the Laplacian in two dimensions,

$$\Delta_{\perp} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (17.11)$$

Typically the velocity boundary conditions are no-slip, i.e.,

$$U(x, y) \equiv v_x(x, y) = 0 \quad \text{and} \quad V(x, y) \equiv v_y(x, y) = 0 \quad \text{on boundary} \quad (17.12)$$

This implies boundary conditions on the first derivative of the streamfunction

$$\partial_x \psi = 0 \quad \text{and} \quad \partial_y \psi = 0 \quad \text{on boundary} \quad (17.13)$$

We now have essentially all the machinery that we need for the problems we are going to deal with. Let us now solve a few examples.

17.3 Planar channel flow

We study Stokes flow in a channel of width $2a$ from left to right.

On the left we impose a pressure of p_{in} and on the right we impose a pressure of p_{out} ; the length of the channel is L . We choose z axis perpendicular to the plane of the channel, x axis along the flow (often called the streamwise direction) and y axis perpendicular to it (often called the wall-normal direction). As this is a planar problem, let us use (17.10).

In terms of the streamfunction the mathematical statement of the problem is as follows. Solve

$$\Delta_{\perp}^2 \psi = 0 \quad (17.14)$$

with boundary condition

$$\partial_x \psi = 0 = \partial_y \psi \quad \text{at} \quad y = \pm a \quad (17.15)$$

Clearly the flow is translationally invariant along the streamwise direction. Hence the bi-Laplacian will reduce to

$$\frac{d^4 \psi}{dy^4} = 0 \quad (17.16)$$

which is solved by

$$\psi(y) = Ay^3 + By^2 + Cy + D. \quad (17.17)$$

which implies

$$U(y) = \partial_y \psi(y) = 3Ay^2 + 2By + C \quad (17.18)$$

With the boundary condition (17.15) we obtain

$$U(y) = A(a^2 - y^2) \quad (17.19)$$

In microfluidic experiment, often not the pressure but the flow-rate is set. In that case, the flow-rate through the channel is

$$\int_{-a}^a U(y) dy = \frac{4A}{3} a^3 \equiv Q, \quad (17.20)$$

which gives $A = 3Q/(4a^3)$. This completes the solution.

17.4 Diverging/converging channel

You may complain that given the long preamble the previous solution is something of a disappointment. It is too simple. We could have done it in our head. Now consider a non-trivial generalization. How to solve this problem in either a converging or a diverging channel? In general, this will be quite a complicated problem. Let us consider a version that we can think about perturbatively. Let the width of the channel change very slowly as a function of x , such that the boundary condition is now

$$U = 0 \quad \text{and} \quad V = 0 \quad \text{at} \quad y = \pm a(x) \quad (17.21)$$

where $a(x)$ is a slow function of x such that there is a dimensionless small parameter $\epsilon \equiv da/dx$. We shall build on the solution we have already found in the previous section. As a is a slow function of x it is now reasonable to assume that x component of velocity satisfies the the following equation

$$U(x, y) = A(x) (a^2(x) - y^2) \quad (17.22)$$

where A and a are no longer constant but functions of x . Here we know the function $a(x)$ – we know the shape of our channel but do not know $A(x)$. However, the flux, Q , still given by

$$Q = \frac{4A}{3} a^3 \quad (17.23)$$

must be a constant of space. Hence we must have

$$A(x) = \frac{3Q}{4a(x)^3} \quad (17.24)$$

Substituting this back in (17.22) we obtain the solution for the velocity $U(x, y)$. We can also calculate the pressure along the streamwise direction by

$$\frac{dp}{dx} = \mu \partial_y^2 U(x, y) = -\mu 2A(x) = -\frac{3\mu Q}{2a^3} \quad (17.25)$$

$$\Rightarrow p(x) = p_{in} - \frac{3\mu Q}{2} \int_0^x \frac{d\xi}{a(\xi)^3} \quad (17.26)$$

17.5 Lubrication approximation

Notice the crucial approximation that helped us solve the problem in the previous section. For a channel whose shape is a slow function of space, we assumed that the flow locally is a parabola as given in Equation (17.19). If instead of a planar channel we had a circular channel then we need to use the problem we solved in Section 15.1.1 where we got

$$Q = \int_0^a v_z(r) 2\pi r dr = \frac{\pi a^4 (p_{in} - p_{out})}{8\mu L} \quad (17.27)$$

This gave a different dependence on a but nothing else would change. This trick is a very useful one if you are solving a problem in Stokes flow where the channel is very long compared to its other two (or one if it is a planar channel) direction. Essentially, the streamwise derivatives can be ignored compared to the wall-normal ones and the wall-normal component of velocity can be ignored compared to the streamwise one.

This approximation will help us understand and describe a wide range of physical phenomenon, from friction in mechanical bearings on one hand to formation/merging of liquid droplets on the other hand.

17.5.1 Tribology

Tribology is the study of friction, wear, lubrication and design of bearings. The study of friction between two solid surfaces – empirical laws of friction first found out by Coulomb is the bedrock of this subject. Remember that the laws of solid friction tells you that the frictional forces are independent of the relative speed between the two surfaces but depends on the normal force, typically the weight of the top object on the bottom one, between them. Now consider a mechanical bearing which we model as a cylinder inside a cylindrical hole, the radius of the inner cylinder is R and the outer one is $R + d$ where $R \gg d$. The inner cylinder rotates inside this gap because of external forces. This is a plain *journal bearing*.

We shall call the inner cylinder shaft/journal and the outer one bearing. We consider a cross-section of the cylinders and reduce the problem to a two-dimensional one. Typically there is some lubricating material in gap. If the friction between the shaft and the bearing obeys laws of solid friction then the moment of the frictional forces – the moment that will try to stop the cylinder from rotating – will be independent of the speed of rotation and will depend linearly on the radius of the shaft. Dimensionally speaking the moment should

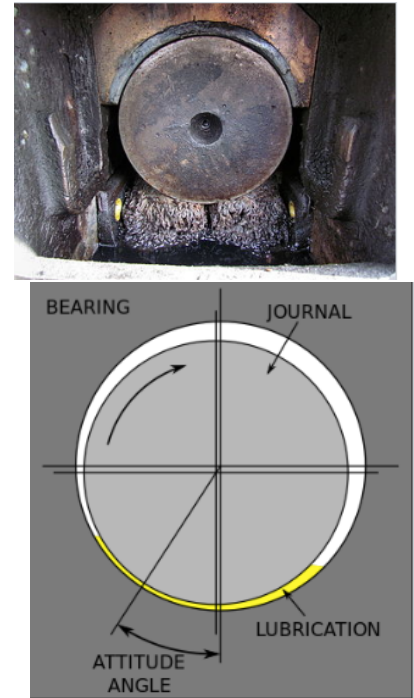


Figure 17.1: Top figure : Plain bearing on a 1906 S-Motor locomotive. Bottom figure: Sketch of cross-section of plain journal bearing. Both the images are from wikipedia.

depend on

$$M \sim \gamma w R \quad (17.28)$$

where γ is the coefficient of solid friction that is dimensionless, w is the weight of the shaft, and R is the radius of the shaft. The other possibility is the following: As the liquid – the lubricating oil – is incompressible there is always a thin liquid layer between the shaft and the bearing and these two surfaces never come into solid contact. The friction is then not dry friction but *wet friction* mediated by this film. The frictional forces should then depend on the viscosity. Dimensionally speaking the moment due to the frictional forces should then have the form

$$M \sim \mu \frac{U_0}{d} R^2. \quad (17.29)$$

The dimension on the left is the moment of a force Newton · m. On the right, we should have viscosity μ , the typical velocity gradient U_0/d where U_0 is the velocity of the surface of the shaft compared to the bearing. To match the dimensions we should then have a factor of m^2 . The characteristic length scale for this should be R . The expressions in (17.28) and (17.29) give very different dependence on U and R . One of the earliest studies² of wet friction was by Russian engineer and military officer N. Petrov way back in 1883.

To work this problem out from Stokes equations let us assume that the shaft is slightly off-center, by an amount e and rotates within the bearing. Let us zoom into the smallest part of the gap between the shaft and the bearing. The gap is wider on one side compared to the other, in the frame of the shaft, the lower surface – that of the bearing – moves with velocity U_0 from left to right. The height of the channel – the height is a function of the horizontal direction, x , – varies with x but slowly. This is already the problem we solved in the previous section except now one of the boundaries moves with $U = U_0$. The streamwise component of velocity as a function of the wall-normal coordinate is given by

$$U = Ay^2 + By + C \quad (17.30)$$

with the boundary condition

$$U = U_0 \quad \text{at} \quad y = 0 \quad (17.31)$$

$$U = 0 \quad \text{at} \quad y = h(x) \quad (17.32)$$

This gives

$$U = U_0 \left(1 - \frac{y}{h}\right) - \frac{1}{2\mu} \left(\frac{dp}{dx}\right) y h \left(1 - \frac{y}{h}\right) \quad (17.33)$$

Here, as in our previous example, dp/dx is not a constant. It is an unknown at this stage. Using the same idea of constancy of flux

¹Do you smell a rat? It seems that it should be m^3 , the problem is that stress is force per unit area, whereas we are considering a two dimensional problem. To make sense we also have to include another length, but this should be unit length perpendicular to the plane of the figure.

²The famous fluid dynamicist Osborne Reynolds did pioneering contribution to the theory of lubrication. The discussion here follows closely the treatment by Sommerfeld, who himself made pioneering contribution to tribology, in his lectures on mechanics of deformable bodies. To quote Sommerfeld, “Unfortunately, Reynolds formulas are difficult to understand, ..”. You may wonder whether our very rough treatment of an engineering problem of great complexity is of any use or not. I shall answer quoting Sommerfeld “One cannot expect an idealizing theory to do better than to establish general rules in an eminently practical problem like ours. Yet these rules may be quite important for practical design as is born out by the success of Mitchell’s segmental bearings”.

A. Sommerfeld. Lectures on theoretical physics : Mechanics of deformable bodies. Levant Books, Kolkata, 2006

through the converging channel we get

$$Q = \int_0^h U(y) dy = \frac{h}{2} U_0 - \frac{h^3}{12\mu} \left(\frac{dp}{dx} \right) \quad (17.34)$$

Remember, this is a converging channel so dp/dx is not a constant but a function of x . It is Q that is a constant. Let us choose h_0 to be the height where dp/dx is zero. This is just a convenient choice. With this I can integrate (17.34) to obtain

$$\frac{h^3}{6\mu} \left(\frac{dp}{dx} \right) = (h - h_0) U_0 \quad (17.35)$$

Now I have obtained dp/dx as a function of the height h and two constant parameters h_0 and U_0 . I can substitute back in (17.33) to obtain $U(x, y)$. Now that I know $U(x, y)$ I can calculate the viscous force on the shaft. This must be given by

$$\sigma_{xy} = \mu \frac{dU}{dy} = -\mu \frac{4h - 3h_0}{h^2} U_0 \quad (17.36)$$

Now transfer this results back to the circular geometry of the journal bearing. The thickness of the gap h is now a function of the angle ϕ given by

$$h = d + e \cos \phi \quad (17.37)$$

We can do a more careful calculation using plane-polar coordinate system but without doing that we can already see, roughly, that the total moment due to the frictional forces going to be

$$M \sim -R(2\pi R)\mu U_0 (J_1 - h_0 J_2). \quad (17.38)$$

Here the first factor of R comes because we are calculating moments due to the viscous force and the other factor of $2\pi R$ comes because we have to integrate over the circumference of the shaft. Here J_1 and J_2 are constants that has emerged from an integration over the angle ϕ from 0 to 2π . They depend on e .

Notice how useful dimensional arguments are. Once we assumed that the friction depends on viscous forces the dimensional arguments gave us the same formula as the more accurate calculation except some geometric factor which hare anyway hard to determine without going into plane-polar coordinate.

17.5.2 Summary

Let me summarize here the key ideas of the lubrication approximation. If you are faced with a problem of flow is narrow channels, (even if the channel has deformable walls), with one dimension of the channel much longer than the other one you should try out the

lubrication approximation. In most cases you may be able to get away with assuming either planar symmetry or axial symmetry although this is not strictly necessary. Here I limit myself to planar symmetry. Take the x axis along the long dimension and y axis along the short dimension. The following approximations together are called the lubrication approximation.

$$\partial_x p \neq 0 \quad \text{and} \quad \partial_y p = 0. \quad (17.39a)$$

$$\partial_y^2 u \gg \partial_x^2 u \quad (17.39b)$$

$$V = 0 \quad (17.39c)$$

These can be derived in a systematic manner using the method of multiple scales. We have derived them using plausible arguments. The equations are appropriate for incompressible flows. Corresponding lubrication approximations for compressible flows, which are relevant while studying collision of droplets in air has also been worked out ³.

3

Exercise 7

1. A circular disk of radius a is parallel to and at a small distance h , $a \gg h$, from a rigid plane, and the space between them is occupied by a fluid. The pressure at the edge of the disk is atmospheric. Show that motion of the disk in the direction normal to the plane gives rise to an opposing force on the disk of the magnitude

$$\frac{3\pi\mu a^4}{2h^3} \frac{dh}{dt} \quad \text{provided} \quad \frac{\rho h}{\mu} \frac{dh}{dt} \ll 1 \quad (17.40)$$

Hence show that a constant force F applied to a disk initially at a distance h_0 will pull it well away from the plane in a time

$$\frac{3\pi\mu a^4}{h_0^2} F. \quad (17.41)$$

The fact that this time is large when h_0 is small is the basis of the phenomenon of viscous adhesion used in 'Scotch tape'. This problem is from Batchelor's book ⁴, section 4.8.

⁴G Batchelor. An Introduction to Fluid Dynamics. Cambridge University Press, Cambridge, UK, 1967

17.A Equation for thin films

Let us now become ambitious and have a look at one of the problems of fluid mechanics that look really difficult. Think about honey dripping from a spoon. The honey forms a long thread at the bottom of which a drop forms. With time this drop gets detached from the rest of the fluid – the drop *breaks up*. Just before breakup the drop

and the long thread connecting it to the spoon may look something like what we show in Fig. 17.A. Can we write down an equation that describes the breakup of this long thread of honey ?

Let us try to do this in the simplest situation. We think of a two dimensional cross-section of the thin film. We take our x axis along the mid plane – in the cross-section the mid plane becomes a “mid line”. The height of the film from the x axis is h . By an evolution equation for the film, we mean an equation for h . There is a systematic way of doing this using the method of multiple scales, see for example the book by Eggers and Fontelos ⁵ for a detailed derivation and further applications. I shall do a rough derivation using ideas we have already used in the lubrication approximation.

Inside the film $\partial_x p$ is non-zero and $\partial_y p = 0$. This is our first assumption. Hence the pressure p is a function of x along. Now for a fixed x coordinate compare the pressure inside and outside the film. The outside pressure is the atmospheric pressure p_{atm} which is a constant. Let the inside pressure be $p_{\text{in}}(x)$. Here I have made it explicit that p_{in} is a function of x . The difference between the two pressure is given by the surface-tension of the film

$$p_{\text{in}} - p_{\text{atm}} = \sigma \frac{\partial^2 h}{\partial x^2}. \quad (17.42)$$

Because the difference in pressure is given by the surface tension, σ , multiplied by the local curvature. In general a surface embedded in three dimension has two principal curvatures. Here we consider a two-dimensional cross-section hence only one curvature is relevant, which is the curvature in this plane. The curvature of a function $h(x)$ is given by its second derivative for small curvature. This is our second approximation. By our first assumption, the pressure inside the film is then given by

$$p(x, y) = p_{\text{in}}(x) = p_{\text{atm}} + \sigma \frac{\partial^2 h}{\partial x^2}. \quad (17.43)$$

Next I assume that $\partial_y^2 u \gg \partial_x^2 u$ where we use $v_x = u(x, y)$ and $v_y = v(x, y)$. From the Stokes equation we then have

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} \quad (17.44)$$

We integrate it with the boundary condition that $u = 0$ at the mid plane $h = 0$. At $y = h$ we obtain

$$u(x, h) = \frac{h^2}{2\mu} \frac{\partial p}{\partial x} = \frac{\sigma}{2\mu} \frac{\partial}{\partial x} \left(\frac{\partial^2 h}{\partial x^2} \right), \quad (17.45)$$

where in the last step we have used (17.43).

5



Figure 17.2: A picture of drop breakup from the book by Eggers and Fontelos.

The evolution equation of h is a conservation equation

$$\partial_t h + \nabla \cdot [h \mathbf{v}(x, h)] = 0 \quad (17.46)$$

where $\mathbf{v}(x, h)$ is the velocity at the boundary of the film x, h . Next we assume that $\mathbf{V} = 0$. This implies

$$\partial_t h + \partial_x [h u(x, h)] = 0. \quad (17.47)$$

Next substitute (17.45) in this to obtain

$$\partial_t h + \frac{\sigma}{2\mu} \frac{\partial}{\partial x} \left(h^3 \frac{\partial^3 h}{\partial x^3} \right) = 0 \quad (17.48)$$

This equation has been studied numerically to study the droplet breakup with appears as singularity of this equation – the breakup implies that h becomes zero at the certain point $x = x^*$ at a finite time $t = t^*$.

17.A.1 A digression on surface tension

From your elementary physics course you have some idea about what surface tension is. Let me give a very brief introduction here. In a liquid every molecule interacts with all its neighboring molecules. Now think of a glass of water. In the bulk of the material every molecule has neighbors which provide a (small compared to a solid) cohesive energy. As this energy stabilizes this gives a negative contribution to total energy. But the unfortunate molecules at the surface have less partners, hence they cannot lower their energy as much as they would like to. So to minimize its energy what a liquid does is to minimize its surface area such that there are least number of molecules at the surface. If the cohesion energy per molecule is E in the bulk, at the surface it will be approximately $E/2$. The surface tension of a liquid is a direct measure of this short-fall of energy per unit area.

For a surface to be at mechanical equilibrium the force arising due to difference of pressure between outside and inside must be balanced by the surface forces. A simple calculation shows that the difference in pressure is balanced by

$$p_{\text{in}} - p_{\text{out}} = \frac{\sigma}{R} \quad (17.49)$$

where R is the local radius of curvature of the film. Actually a surface at every point has two principal radii of curvature, but as we are going to consider thin films in two dimension we have to consider only one radius of curvature – the other one is infinitely large.

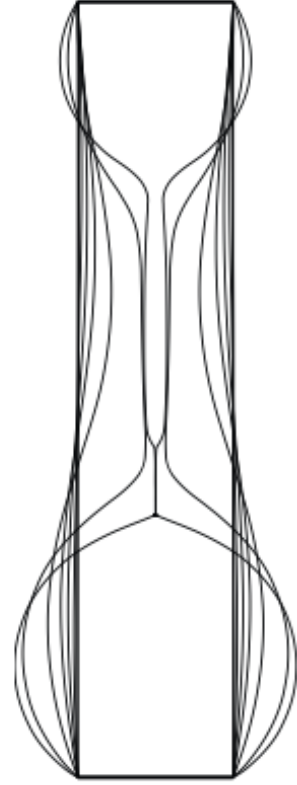


Figure 17.3: Numerical solutions of (17.48), from the book by Eggers and Fontelos.

Surface Tension: A beautiful introduction to surface phenomenon can be found in a book wonderfully named as “Capillarity and Wetting Phenomena: Drops, bubbles, pearls, waves” . My discussion here largely borrowed from this book. The argument about the shortfall of surface energy as a measure of surface tension is more than an appealing picture it even gives the numerical value of surface tension in the right ball-park. For example in water the surface tension is about 72 mN/m.

18 Creeping flow due to moving objects

18.1 Axial symmetry

Clearly there can be no incompressible flow problem that is fully spherically symmetric. We need either a velocity or a pressure gradient to break the spherical symmetry. The most symmetric case we can have in three dimensions is axial symmetry. For example, in the case of a flow around a sphere that is moving with a constant velocity, the velocity of the sphere breaks the spherical symmetry. But the problem still has axial symmetry. Thus even in full generality we can assume that there is always one vector $\mathbf{a}$ that sets the symmetries of the problem. Of course the problem could be more complicated. For example we could have an arbitrarily shaped body (no rotational symmetry) moving with a constant velocity. The problem possesses no symmetry at all, but still the velocity, denoted by the vector $\mathbf{a}$ in one of the quantities that define the problem.

We make the following ansatz

$$v_\alpha = a_\beta \mathcal{U}_{\alpha\beta} \psi \quad (18.1)$$

where $\mathcal{U}_{\alpha\beta}$ is a second-order tensor which is also a differential operator, operating on the function ψ given by

$$\mathcal{U}_{\alpha\beta} = \left(\delta_{\alpha\beta} \frac{\partial^2}{\partial x_\mu \partial x_\mu} - \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \right) \quad (18.2)$$

where the function ψ is, in general a function of all the coordinates.

How did I come up with ansatz ? The argument runs as follows:

1. The problem is linear, hence the solution, both the velocity and the pressure must be proportional to the vector $\mathbf{a}$ and no higher power of $\mathbf{a}$.
2. The velocity must be a vector (a first rank tensor), so it can be in general a scalar function multiplied by $\mathbf{a}$, so it could be

$$v_\alpha = a_\alpha \psi \quad (18.3)$$

where ψ is a scalar function of coordinates x_1, x_2 and x_3 . What we write on (18.1) is a more general form.

Here I follow closely the book on mechanics of swimming and flying by S. Childress .

Stephen Childress. *Mechanics of swimming and flying*, volume 2. Cambridge University Press, 1981

3. Could we have lower order derivatives in (18.2), for example, could we have

$$v_\alpha = \epsilon_{\alpha\beta\gamma} \partial_\beta \chi(x_1, x_2, x_3) a_\gamma? \quad (18.4)$$

Not if $\mathbf{a}$ is a polar vector (a vector that changes sign under parity inversion), because then the left and right hand side of (18.4) will have different symmetries under parity. This would be a possibility if $\mathbf{a}$ where an axial vector, e.g., an angular velocity, but not an usual velocity.

4. Could we have another term in (18.2) of the following form ?

$$v_\alpha = \epsilon_{\alpha\beta\gamma} a_\beta \epsilon_{\gamma\mu\nu} \partial_\mu \partial_\nu \psi \quad (18.5)$$

This is allowed by symmetry, but the rules of produces of $\epsilon_{\alpha\beta\gamma}$ reduces this to (18.2).

5. Could we have terms involving third order derivatives in (18.2) ? For example

$$v_\alpha = \epsilon_{\alpha\beta\gamma} a_\mu \partial_\beta \partial_\gamma \partial_\mu \psi? \quad (18.6)$$

We shall land up with the same objection about parity as in item 3 above.

6. What about fourth order derivatives ? For example,

$$v_\alpha = a_\mu \partial_\beta \partial_\beta \partial_\mu \partial_\alpha \psi? \quad (18.7)$$

This is certainly allowed. But it is more complicated than (18.2).

7. You can see that any odd order derivatives of a scalar function is allowed if the vector $\mathbf{a}$ is a axial vector and any even order derivative is allowed if the vector $\mathbf{a}$ is a polar vector. The simplest prescriptions – by that I imply one that contains the least number of necessity derivatives – is given by (18.4) in the first case and (18.2) in the second case.

This choice for $\mathcal{U}_{\alpha\beta}$ is such that we identically satisfy the incompressibility constraint. Next we put an ansatz for the pressure

$$p = \mu a_\beta \Pi_\beta \quad (18.8)$$

where Π is a vector that is undetermined as this point. Substituting (18.2) and (18.8) in the Stokes equation

$$\mu \nabla^2 \mathbf{v} = \nabla p, \quad (18.9)$$

we obtain

$$a_\beta \mu [\delta_{\alpha\beta} \partial_{\gamma\gamma} \partial_{\mu\mu} \psi - \partial_\alpha \partial_\beta \partial_{\gamma\gamma} \psi] = \mu a_\beta \partial_\alpha \Pi_\beta \quad (18.10)$$

This can be satisfied, if we choose

$$\partial_{\nu\nu}\partial_{\mu\mu}\psi = 0 \quad (18.11)$$

and

$$\Pi_\beta = -\partial_\beta\partial_{\nu\nu}\psi \quad (18.12)$$

As uniqueness guarantees that once I have found a solution, it is *the* solution I am happy with the state of affairs now. This also implies that I do not have to consider the other possibilities allowed by symmetry that contains higher order derivatives. Once I find ψ by solving (18.11) I can obtain the pressure. The function ψ is a scalar function but it is typically not called streamfunction as its contour lines do not give the streamlines any more. The boundary conditions appear on second order derivatives of ψ .

18.2 Sedimenting sphere

As an example of a Stokes problem with axial symmetry let us consider a sedimenting sphere. We know that under the drag force the sphere will reach a terminal velocity. Before we start calculating, let us see how much dimensional analysis can help us. Clearly the drag force should depend on the velocity of the sphere, but at what power? Remember that the Stokes equation is linear, so the drag force should be proportional to the velocity. Next, it should depend on viscosity, because the drag is a viscous force. Dimensionally the only form a force can take is then

$$F_{\text{drag}} \sim \mu U (\text{length}) \quad (18.13)$$

The only choice of the length scale is the radius (or diameter, there is no distinction between them as far as dimensional analysis is concerned), hence we should get

$$F_{\text{drag}} \sim (\text{constant}) \mu U R \quad (18.14)$$

where R is the radius of the sphere. Notice an interesting fact. If you imagined that the drag appears because molecules in the fluid move and hit the sphere and thus transfer momentum to it in a way similar to how we calculated the pressure of an ideal gas then the drag force should be proportional to not the radius but the surface area of the sphere which is not what we get in the continuum picture.

Now let us get back to solving the boundary value problem to see how much better we can do with the most accurate calculation. We are solving the Stokes equation with axisymmetry, hence we use the formulation in section 18.1. To obtain a solution we first solve the bi-Laplacian equation for a scalar potential.

$$\nabla^2 \nabla^2 \psi = 0 \quad (18.15)$$

Let us assume that ψ is a function of the radial coordinate r alone. Let us use units of length such that $R = 1$. The sphere moves with constant velocity vector $\mathbf{U}$. Solving the bi-Laplacian equation and setting no-slip boundary condition I can show that the velocity and pressure outside the sphere is given by

$$v_\alpha = U_\alpha - \frac{3}{4} \left(U_\beta \frac{x_\alpha x_\beta}{r^3} + U_\alpha \frac{1}{r} \right) - \frac{1}{4} \frac{\partial}{\partial x_\alpha} \left(\frac{U_\beta x_\beta}{r^3} \right) \quad (18.16a)$$

$$p = -\mu \frac{3}{2} \frac{U_\beta x_\beta}{r^3} \quad (18.16b)$$

where the vector $\mathbf{r} = (x_1, x_2, x_3)$ and consequently $r^2 = x_\alpha x_\alpha$. The solution is valid for $r > 1$. The details of the calculation is left as an exercise.

Exercise 8

1. **Stokes solution for sedimenting sphere** Following section 18.2 prove (18.16). Do not omit any intermediate steps. Particularly show how you get the boundary condition at $r = 1$ which corresponds to the surface of the sphere as we have normalized our distances with the radius of the sphere R .
2. **Galilean invariance** Show by explicit transformation that the Stokes equation is Galilean invariant.

18.3 Green's function: the stokeslet

It is instructive to study the solution (18.16). It is written down in the frame of the sphere which is moving with velocity $\mathbf{U}$. That is why the speed at infinity is finite, $\mathbf{U}$. Going to a coordinate system moving with a constant velocity is a Galilean transformation. Newton's laws are invariant under Galilean transformation, we started with them, so what we have obtained – the Stokes and also the Navier–Stokes equation are also Galilean invariant. So we can move back to the laboratory frame and write down the solution of the flow outside a moving sphere, moving with a velocity $\mathbf{U}$ to be the velocity field

$$v_\alpha = -\frac{3}{4} \left(U_\beta \frac{x_\alpha x_\beta}{r^3} + U_\alpha \frac{1}{r} \right) - \frac{1}{4} \frac{\partial}{\partial x_\alpha} \left(\frac{U_\beta x_\beta}{r^3} \right) \quad (18.17)$$

and a pressure field as

$$p = -\mu \frac{3}{2} \frac{U_\beta x_\beta}{r^3} \quad (18.18)$$

As $r \rightarrow \infty$, the pressure field goes to zero as $1/r^2$. As the pressure is the solution of Laplace equation, this behaviour is expected. The velocity field goes to zero as $1/r$ as $r \rightarrow \infty$.

To see this better let us go back to (18.11). Any ψ that satisfies the bi-harmonic equation gives a corresponding solution of the Stokes equation. Roughly speaking, the velocity is two derivatives on ψ . If we want the velocity to vanish at infinity at most as $1/r$ then we can consider

$$\psi^{\text{stokeslet}} = \frac{r}{8\pi} = \frac{1}{8\pi} \sqrt{x_1^2 + x_2^2 + x_3^2} \quad (18.19)$$

I can find the velocity and the pressure of such a streamfunction through (18.2) and (18.12) respectively, let me call them $\mathbf{v}^{\text{stokeslet}}$ and $p^{\text{stokeslet}}$ respectively. We expect the velocity to go as $\sim 1/r$ and the pressure to go as $\sim 1/r^2$ which is the same dependence at large r that we got for the sphere. Using (18.2) and (18.8) we obtain

$$\nabla^2 \mathbf{v}^{\text{stokeslet}} - \nabla p^{\text{stokeslet}} = -\mathbf{a} \nabla^2 \nabla^2 \psi^{\text{stokeslet}} = -\mathbf{a} \nabla^2 \left(\frac{1}{4\pi r} \right) = \mathbf{a} \delta(\mathbf{r}) \quad (18.24)$$

where $\delta(\cdot)$ is the three dimensional Dirac delta function. The velocity $\mathbf{v}^{\text{stokeslet}}$ and the pressure $p^{\text{stokeslet}}$ are explicitly given by

$$\mathbf{v}_\alpha^{\text{stokeslet}} = \mathcal{U}_{\alpha\beta} a_\beta \psi^{\text{stokeslet}} = \frac{1}{8\pi} \left(\frac{\delta_{\alpha\beta}}{r} + \frac{x_\alpha x_\beta}{r^3} \right) a_\beta \quad (18.25a)$$

$$p^{\text{stokeslet}} = \Pi_\beta a_\beta \psi^{\text{stokeslet}} = \frac{1}{4\pi} \left(\frac{x_\beta}{r^3} \right) a_\beta \quad (18.25b)$$

This is a very useful result. To illustrate its significance, let me consider the Stokes equation with a force on its right hand side. To start with let us think of the simplest force – one that is concentrated at one point in space, $\mathbf{r} = \mathbf{r}^0$ and its value at $\mathbf{r}^0$ is $\mathbf{a}$, in other words, let us consider a force,

$$\mathbf{F}(\mathbf{r}) = \mathbf{a} \delta(\mathbf{r} - \mathbf{r}^0) \quad (18.26)$$

on the RHS of the Stokes equation. Let us perform a coordinate translation such that the origin of our coordinate system is moved to $\mathbf{r}^0$. In this coordinate system the inhomogeneous Stokes equation with a force $\mathbf{F}$ given by (18.26) is

$$\nabla^2 \mathbf{v} - \nabla p = \mathbf{a} \delta(\mathbf{r}) \quad (18.27)$$

This is precisely the same equation satisfied by $\mathbf{v}^{\text{stokeslet}}$ and $p^{\text{stokeslet}}$. Going back to the coordinate where the problem was originally posed, we can simply write down the solution of the partial differential equation (18.27) as

$$\mathbf{v}_\alpha(\mathbf{r}) = \frac{1}{8\pi} \left(\frac{\delta_{\alpha\beta}}{|\mathbf{r} - \mathbf{r}^0|} + \frac{(x_\alpha - x_\alpha^0)(x_\beta - x_\beta^0)}{|\mathbf{r} - \mathbf{r}^0|^3} \right) a_\beta \quad (18.28a)$$

$$p(\mathbf{r}) = \frac{1}{4\pi} \left(\frac{x_\beta - x_\beta^0}{|\mathbf{r} - \mathbf{r}^0|^3} \right) a_\beta \quad (18.28b)$$

Why the factor of 8π ?

This is because how I choose to normalize the three-dimensional delta function. I choose to define it by

$$\int \delta(\mathbf{r}) dV = 1, \quad (18.20)$$

where the integral is over all space. An integration over the full solid angle in the integration above gives a factor of 4π , so I get

$$4\pi \int_0^\infty dr \delta(r) = 1. \quad (18.21)$$

The delta function above is an one dimensional delta function, but from 0 to infinity. The usual definition of one-dimensional delta function is

$$\int_{-\infty}^\infty \delta(x) dx = 1. \quad (18.22)$$

By construction, the delta function is even, hence

$$\int_0^\infty dr \delta(r) = \frac{1}{2}. \quad (18.23)$$

Now let us consider a slightly more complicated problem – the force $\mathbf{F}(\mathbf{r})$ is now concentrated at many discrete points $\mathbf{r}^j$ in each of which its value is $\mathbf{f}^j$, i.e.,

$$\mathbf{F}(\mathbf{r}) = \sum_j \mathbf{f}^j \delta(\mathbf{r} - \mathbf{r}^j). \quad (18.29)$$

As the Stokes problem is linear, to obtain the solution to this problem we use the solution to the problem (18.27), in particular the solution is

$$v_\alpha(\mathbf{r}) = \sum_j \frac{1}{8\pi} \left(\frac{\delta_{\alpha\beta}}{|\mathbf{r} - \mathbf{r}^j|} + \frac{(x_\alpha - x_\alpha^j)(x_\beta - x_\beta^j)}{|\mathbf{r} - \mathbf{r}^j|^3} \right) f_\beta^j \quad (18.30a)$$

$$p(\mathbf{r}) = \sum_j \frac{1}{4\pi} \left(\frac{x_\beta - x_\beta^j}{|\mathbf{r} - \mathbf{r}^j|^3} \right) f_\beta^j \quad (18.30b)$$

Now to the most general problem. The force $\mathbf{F}$ is nonzero on a region of space. For example, consider a body of general shape moving in a fluid. You can imagine that from every point of this body a force acts on the fluid. What is the flow velocity and pressure at point $\mathbf{r}$ in the flow? We can find the solution by now considering $\mathbf{r}^j$ as a continuous variable (instead of a discrete one considered so far) – we call this $\mathbf{r}^s$. The force $\mathbf{f}^j(\mathbf{r}^j)$ now become the function of the same continuous variable $\mathbf{r}^s$ and the sum in (18.30) becomes an integral,

$$v_\alpha(\mathbf{r}) = \int d\mathbf{r}^s \frac{1}{8\pi} \left[\frac{\delta_{\alpha\beta}}{|\mathbf{r} - \mathbf{r}^s|} + \frac{(x_\alpha - x_\alpha^s)(x_\beta - x_\beta^s)}{|\mathbf{r} - \mathbf{r}^s|^3} \right] f_\beta(\mathbf{r}^s) \quad (18.31a)$$

$$p(\mathbf{r}) = \int d\mathbf{r}^s \frac{1}{4\pi} \left[\frac{x_\beta - x_\beta^s}{|\mathbf{r} - \mathbf{r}^s|^3} \right] f_\beta(\mathbf{r}^s) \quad (18.31b)$$

Let me now reframe the problem in the more abstract nomenclature of general linear partial differential equations. The Stokes equation is a linear, homogeneous partial differential equation (PDE) of second order. You can add a “delta-function source” to its right hand side to turn it into an inhomogeneous PDE. This is (18.27) with $|\mathbf{a}| = 1$. The solution to this problem is called the Green’s function of the inhomogeneous PDE. So the Green’s function to the inhomogeneous Stokes equation are (18.25). The Green’s function is a function of both the source coordinate $\mathbf{r}^s$ and the field coordinate $\mathbf{r}$. For (linear) homogeneous PDEs the Green’s function must be a function of $\mathbf{r} - \mathbf{r}^0$. To obtain the solution of the general inhomogeneous problem – one with a “general source” on the RHS of the PDE – I multiply the Green’s function by the “strength of the source”, $f(\mathbf{r}^s)$ and integrate over the “source coordinate” $\mathbf{r}^s$. This is (18.31). We now write this in

more compact form as

$$v_\alpha(\mathbf{r}) = \int d\mathbf{r}^s G_{\alpha\beta}(\mathbf{r} - \mathbf{r}^s) f_\beta(\mathbf{r}^s) \quad (18.32a)$$

$$p(\mathbf{r}) = \int d\mathbf{r}^s H_\beta(\mathbf{r} - \mathbf{r}^s) f_\beta(\mathbf{r}^s) \quad (18.32b)$$

with the velocity and pressure Green's function give by

$$G_{\alpha\beta}(\mathbf{r} - \mathbf{r}^s) = \frac{1}{8\pi} \left[\frac{\delta_{\alpha\beta}}{|\mathbf{r} - \mathbf{r}^s|} + \frac{(\chi_\alpha - \chi_\alpha^s)(\chi_\beta - \chi_\beta^s)}{|\mathbf{r} - \mathbf{r}^s|^3} \right] \quad (18.33a)$$

$$H_\beta(\mathbf{r} - \mathbf{r}^s) = \frac{1}{4\pi} \left[\frac{\chi_\beta - \chi_\beta^s}{|\mathbf{r} - \mathbf{r}^s|^3} \right] \quad (18.33b)$$

The velocity Green's function is a second rank tensor and the pressure Green's function is a first rank tensor (vector).

Notice that the velocity Green's function falls off very slowly at large relative distance, $r \equiv |\mathbf{r} - \mathbf{r}^0|$ as $\sim \frac{1}{r^2}$ – an effect of a localized disturbance in the flow can be felt at very large distance away.

18.3.1 Electrostatic analogy

You may have already seen Green's functions without calling them as such. From electrostatics you know that the potential of a point charge, e , at the origin satisfies the equation

$$\nabla^2 \Phi = -e\delta(\mathbf{r}) \quad (18.34)$$

The potential satisfied the Laplace equation everywhere except at the origin where the charge is. The point charge is the “delta-function source” for the Laplace's equation. If you have a charge density $\rho(\mathbf{r}^s)$ as a function of the source coordinate $\mathbf{r}^s$ then, by usual superposition principle, the potential at a target point $\mathbf{r}$ is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int d\mathbf{r}^s \frac{\rho(\mathbf{r}^s)}{|\mathbf{r} - \mathbf{r}^s|} \quad (18.35)$$

The Green's function of the Laplace's equation is then given by

$$G_{\text{laplace}}(\mathbf{r} - \mathbf{r}^s) = \frac{1}{4\pi} \frac{1}{|\mathbf{r} - \mathbf{r}^s|} \quad (18.36)$$

If we consider the electric field of charge distribution, it is given by

$$E_\alpha(\mathbf{r} - \mathbf{r}^s) = \frac{1}{4\pi\epsilon_0} \int d\mathbf{r}^s \frac{\chi_\alpha - \chi_\alpha^s}{|\mathbf{r} - \mathbf{r}^s|^3} \rho(\mathbf{r}^s) \quad (18.37)$$

The corresponding Green's function is

$$G_\alpha(\mathbf{r} - \mathbf{r}^s) = \frac{1}{4\pi\epsilon_0} \frac{\chi_\alpha - \chi_\alpha^s}{|\mathbf{r} - \mathbf{r}^s|^3} \quad (18.38)$$

The source in all these cases is a scalar source – charge density. The source in Stokes equation is a vector source – a force. The simplest vector source in electrostatics is a dipole. The electrostatic potential due to a dipole $\mathbf{p}$ at $\mathbf{r}^s$ is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p}_\alpha(x_\alpha - x_\alpha^s)}{|\mathbf{r} - \mathbf{r}^s|^3}. \quad (18.39)$$

This is precisely the Green's function of pressure in the Stokes problem. So the pressure in the Stokes equation is the same as the potential of a dipole.

18.4 Drag

As we know the flow everywhere we can now calculate the force exerted by the fluid on a sphere moving with a velocity $\mathbf{U}$. Clearly this is the force that is necessary to keep the sphere moving. Before we do this let us try to understand this calculation in the simplest context. Think about a rigid boundary of a steady flow. If we zoom in very close to the surface we can use a cartesian coordinate system whose x direction points along the direction of the flow just above the surface, z direction is the normal to the surface and the y direction is along the surface perpendicular to both x and z . We denote the three components by U, V, W . At the surface all three components of velocity are zero. But all the derivatives are not zero. As all the components are zero at $z = 0$ – the surface – for all values of x, y ,

$$\partial_x U = \partial_x V = \partial_x W = 0 \quad (18.40)$$

$$\partial_y U = \partial_y V = \partial_y W = 0 \quad (18.41)$$

The way we have formulated our problem, if we zoom close to the surface (but not on it) $V = 0$ everywhere, so

$$\partial_z V = 0, \quad (18.42)$$

on the surface. Furthermore the incompressibility constraint must be satisfied, so

$$\partial_x U + \partial_z W = 0 \quad (18.43)$$

By virtue of (18.40) $\partial_z W = 0$. The strain tensor at the surface then has the following form

$$S_{\alpha\beta} = \begin{pmatrix} 0 & 0 & \partial_z U \\ 0 & 0 & 0 \\ \partial_z U & 0 & 0 \end{pmatrix} \quad (18.44)$$

The stress by the fluid on the boundary is given by

$$\sigma_{\alpha\beta} = \mu S_{\alpha\beta}. \quad (18.45)$$

Only two components of the stress survives. What is the component of the force (by the fluid on the surface) along the x direction ? The component of the force along the x direction on a surface whose normal is along the z direction is given by

$$\sigma_{xz} = \mu S_{xz} = \mu \frac{\partial U}{\partial z} \quad (18.46)$$

We can now generalize this. For a point on any surface we can define an unit normal¹, which we denote by $\hat{n}$ and use the symbol x_n for the coordinate direction along it. To define a Cartesian coordinate system we now need two tangents on the surface. These are not unique, there are infinite number of possibilities on the surface. We choose the one in which one direction points towards the direction of the flow just above the surface; we call this x_t with unit vector $\hat{t}$. Then the force due to the flow on a small element of area ΔA on the surface is only along $\hat{t}$ and is given by

$$F_t = \sigma_{tn} \Delta A = \mu S_{tn} \Delta A \quad (18.47)$$

I avoid doing a detailed calculation here. It is moved to section ??.

Let me mention the final result: the drag force on a sphere of radius a moving with a constant velocity U_0 is

$$F_{\text{drag}} = 6\pi\mu a U_0 \quad (18.48)$$

18.5 Swimming at low Reynolds number

Now consider an organism trying to swim in a very viscous fluid. For a bacteria or sperm water is a very viscous fluid. I hope you have understood by now that *inertia-less* means. The flow is determined by the *instantaneous* configuration of the boundaries. Now think about a bacterial going through a certain motion to swim. To be able to keep doing this, the process must go on cycle, i.e., the initial and final configuration must be the same. Now assume that the organism is very simple, so simple that it has only one movable hinge; like a scallop that can move only by opening and closing one valve. Then over one cycle the valve opens through a series of configuration and closes through exactly the same series of configurations. So the flow goes through exactly the same velocities and pressure. Hence the drag on the organism – which is actually a push that make the organism move – is zero over a full cycle. A scallop cannot swim low Reynolds number. It does not matter if it opens the valve at a rate faster than it closes it, the speed does not matter as the fluid practically instantaneously adjusts to the changed boundary conditions. This is Purcell's "Scallop Theorem". Of course, actually scallops have no problem

¹Well, almost always. If the surface has a corner we cant do that. But let us consider only smooth surfaces here.

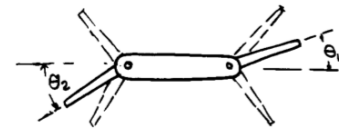


Figure 18.1: Purcell's minimal swimmer. Figure taken from "Life at low Reynolds number" by E.M. Purcell.

swimming, they are must larger than bacteria, they swim at non-zero Reynolds number. How complicated an organism have to be for it to be able to be a swimmer at low Reynolds number ? It must be able to go over a cycle without repeating the same configuration. For the scallop the number of degrees of freedom is just one angle. With two angles it is possible to swim – this is Purcell’s minimal swimmer.

18.6 *Mixing*

The linearity of the Stokes problem and its instantaneous nature has implication for another crucial property – mixing. This is best seen in the famous demonstration by G.I. Taylor. There is a youtube video <https://www.youtube.com/watch?v=51-6QCJTAjU> of the great man giving this demonstration, I insist that you watch this video. He has an apparatus with two concentric cylinders, the annulus between them is filled up with glycerine. The inner cylinder can be rotated by rotating a handle. He puts a drop of dye in the glycerine which stays put. Then he rotates the inner cylinder clockwise four times and it seems that the dye is mixed, he then rotates it four times counter-clockwise and the lo and behold the spot of dye is back with a little fuzziness due to molecular diffusion ! There is no mixing.



Figure 18.2: G.I. Taylor with his apparatus. A still from the youtube video at <https://www.youtube.com/watch?v=51-6QCJTAjU>.

19 Viscosity of suspension

19.1 Pollens in water

Suspensions are quite common in nature, for example, a cloud is a suspension of water droplets in air. There are wide varieties of suspensions, of one liquid phase in another (mayonese), several different objects in water (curry), deformable objects in water (blood) the possibilities are endless. In this chapter I am going to introduce you to calculation of effective viscosity of suspensions. Let us start by considering the simplest problem, that of rigid objects in a background liquid a model that I call “pollens in water”. For such a suspension to make sense the density of the rigid objects and the background fluid must be quite close otherwise the objects would precipitate under gravity. The problem can be solved completely only in the limit where the number density of the suspended objects is small and the objects themselves are small spheres. This is the problem that I intend to solve now.

Let me now be more precise in formulating the problem. A typical device to measure viscosity, is a rotating viscometer which is essentially a Taylor-Couette apparatus – two coaxial cylinders with a gap between them. The outer cylinder is rotated with a constant angular velocity Ω_b , but the inner cylinder is held fixed. The torque necessary to hold it fixed is measured. From this torque the viscosity of the material in between the two cylinders can be calculated. Conceptually, you can also think about two parallel walls with the fluid in between them. One of the walls are moved with a constant velocity U_0 and the force on the other wall, which is held fixed, is measured. This setup is called “plane Couette flow”.

Now let us try a “thought-experiment”, we measure the viscosity of a suspension by introducing it in a plane Couette apparatus. If we do a Galilean transformation we can move to the midplane of the apparatus. In this frame we shall see the top wall going to the right with a velocity $\hat{x} \frac{U_0}{2}$ and the bottom wall going to the left with a velocity $-\hat{x} \frac{U_0}{2}$. Let this width of the apparatus be d . The velocity

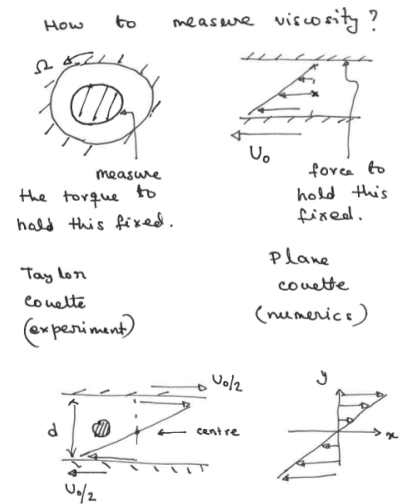


Figure 19.1: A cartoon of a ‘thought-experiment’ of measurement of viscosity of a suspension.

gradient tensor of this flow is

$$B_{\alpha\beta} = \partial_\beta v_\alpha = \begin{pmatrix} 0 & \dot{\gamma} \\ 0 & 0 \end{pmatrix} \quad (19.1)$$

where $\dot{\gamma} = U_0/d$. You can decompose the velocity gradient matrix to its symmetric and anti-symmetric counterpart

$$\begin{pmatrix} 0 & \dot{\gamma} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\dot{\gamma}}{2} \\ \frac{\dot{\gamma}}{2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{\dot{\gamma}}{2} \\ -\frac{\dot{\gamma}}{2} & 0 \end{pmatrix} \quad (19.2)$$

The anti-symmetric part is a pure rotation which introduces no deformation. The symmetric part is the one that matters when we consider viscosity. Hence in rest of this discussion we are going to ignore the anti-symmetric part. We then land up with rigid spheres in a background flow with a velocity gradient matrix that is a pure strain everywhere. Let us take a break from solving the effective viscosity of the suspension and solve this problem first.

19.2 One rigid sphere in a strain

To be precise, we consider the following problem. A rigid sphere of radius a is at the origin. There is a background flow

$$v_\alpha^0 = S_{\alpha\beta}^0 x_\beta \quad (19.3)$$

where S^0 is a symmetric tensor with zero trace – because v^0 must be divergenceless. Furthermore this background flow has constant pressure everywhere. Without any loss of generality we choose this constant to be zero. This is what the flow looks like at large distance away from the sphere – at infinity. Find out the flow everywhere.

This is similar to the problem we solved while considering a rigid sphere moving with a constant velocity. In that case, once we moved to the body-fixed frame of the sphere we had to solve for a flow that becomes a constant at large distance away from the sphere. In the present case the flow is not a constant in space but given by (19.3).

At present we know no method that allows us to solve this flow for all Reynolds number. But we can solve it for flows with very small Reynolds number – essentially Stokes flow. Let us assume that the solution to the Stokes problem can be written as the sum of two functions

$$\mathbf{v} = \mathbf{v}^0 + \mathbf{w} \quad (19.4)$$

where $\mathbf{w}$ must be a vector function that goes to zero at infinity. Obviously $\mathbf{w}$ depends on $S_{\alpha\beta}^0$. This is allowed only for solutions of Stokes equation – not the Navier–Stokes equation – because the Stokes equation is linear.

Next, let us see how well the symmetry arguments work in this case.

1. The solution depends on the tensor $S_{\alpha\beta}^0$. As the problem is linear the solution must depend linearly on $S_{\alpha\beta}^0$.
2. The solution must have the proper tensorial character of a vector, so one possibility is

$$w_\alpha \propto S_{\alpha\beta}^0 x_\beta \quad (19.5)$$

which cannot be the case because this solution does not go to zero at infinity. Taking cue from how we constructed the solution to the Stokes problem, we realise that we need three derivatives on a scalar function ψ . Then we pairwise contract four indices among the five indices (three indices from the derivatives and two from the $S_{\alpha\beta}^0$ to obtain one index of the vector. So we start with

$$w_\alpha = \left(\begin{array}{l} \text{contract 4 indices,} \\ \text{set the remaining} \\ \text{to } \alpha \end{array} \right) S_{\gamma\beta}^0 \frac{\partial^3}{\partial x_\mu \partial x_\nu \partial x_\lambda} \psi \quad (19.6)$$

and then contract. I can contract μ with either ν or λ . One way to do this is to add one more gradient to the operator $\mathcal{U}_{\mu\nu}$ that we considered in the last chapter, i.e.,

$$w_\alpha = \left(\begin{array}{l} \text{contract 4 indices,} \\ \text{set the remaining} \\ \text{to } \alpha \end{array} \right) S_{\gamma\beta}^0 \frac{\partial}{\partial x_\lambda} \mathcal{U}_{\mu\nu} \psi \quad (19.7)$$

Remember that $\nabla \cdot \mathbf{w} = 0$. This is already assured by the form of the function we have chosen, in particular

$$\begin{aligned} w_\alpha &= \mathcal{U}_{\alpha\beta} S_{\beta\lambda}^0 \partial_\lambda \psi \\ &= \left(\delta_{\alpha\beta} \frac{\partial^2}{\partial x_\mu \partial x_\mu} - \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \right) S_{\beta\lambda}^0 \partial_\lambda \psi \\ &= S_{\beta\lambda}^0 \left(\delta_{\alpha\beta} \frac{\partial^2}{\partial x_\mu \partial x_\mu} - \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \right) \left(\frac{\partial}{\partial x_\lambda} \right) \psi \end{aligned} \quad (19.8)$$

3. In a way similar to the way we constructed the pressure in the Stokes problem we write

$$p = S_{\alpha\beta}^0 \mathcal{P}_{\alpha\beta} \psi \quad (19.9)$$

substitute this back into the Stokes equation to obtain

$$\nabla^2 \nabla^2 \psi = 0 \quad (19.10)$$

and

$$\mathcal{P}_{\alpha\beta} = \left(\frac{\partial^2}{\partial x_\alpha \partial x_\beta} \right) \nabla^2 \psi \quad (19.11)$$

4. We know the simplest ψ that satisfies the condition that the velocity goes to zero at infinity,

$$\psi \propto \psi^{\text{Stokeslet}} = Ar \quad (19.12)$$

where A is a constant – it was $1/8\pi$ in the Stokeslet. But there is another boundary condition at $r = a$, to satisfy that we need at least one more constant. So we must have

$$\psi(r) = Ar + \frac{B}{r} \quad (19.13)$$

5. So the solution to the problem we posed is given by (19.8) and (19.14b) together with (19.11) and (19.13). After some algebra that is left as an exercise, you will get the solution to be

$$w_\mu = \frac{5}{2} \left(\frac{a^5}{r^4} - \frac{a^3}{r^2} \right) S_{\alpha\beta}^0 \frac{x_\mu x_\alpha x_\beta}{r^3} - \frac{a^5}{r^4} S_{\mu\nu}^0 \frac{x_\nu}{r} \quad (19.14a)$$

$$p = -5\mu \frac{a^3}{r^3} S_{\alpha\beta}^0 \frac{x_\alpha x_\beta}{r^2} \quad (19.14b)$$

This may look apparently wrong because the pressure is negative, but remember that we choose our scale of pressure such that the background flow had a constant pressure everywhere which we set to zero. So the result is not unphysical, it just means that in a flow like this as you move towards the solid sphere from infinity pressure decreases roughly as $1/r^3$ with the minimum pressure appearing on the surface of the sphere $r = a$.

19.3 Back to suspensions

Back after the break, let us get back to our suspension. We have many rigid spheres in a suspension. I hope I have convinced you that instead of doing an actual experiment of measuring the viscosity of this suspension we can consider a thought-experiment where this suspension has a background flow that has no vortical part, $S_{\alpha\beta}^0$. Due to the presence of the particles, the flow is not going to be simply $S_{\alpha\beta}^0 x_\beta$, but something more complicated. That flow is going to have a stress tensor $\sigma_{\alpha\beta}$. Let us now imagine that we have replaced the suspension by a pure Newtonian fluid with viscosity μ_* . This fluid has a rate-of-strain $S_{\alpha\beta}^0$. The stress tensor of this fluid is going to be $\mu_* S_{\alpha\beta}^0$. Let us demand that the viscosity μ_* is such that the stress tensor of this Newtonian fluid is going to be equal to the stress tensor of the suspension *in a volume-averaged sense*, i.e.,

$$\frac{1}{V} \int_V \sigma_{\alpha\beta} dV = \mu_* S_{\alpha\beta}^0 \quad (19.15)$$

Equation (19.15) is the definition of the effective viscosity μ_* .

Let us now try to calculate the integral on the left of (19.15). The volume of integration is sketched in Fig. 19.3. To stress tensor can be decomposed into two parts. One due to the background flow whose strain-rate is $S_{\alpha\beta}^0$. This is the only contribution to the stress if there were no pollens. This background flow is changed due to the presence of the pollens – we call the additional stress due to this change to be $\sigma'_{\alpha\beta}$;

$$\sigma_{\alpha\beta} = \mu S_{\alpha\beta}^0 + \sigma'_{\alpha\beta}, \quad (19.16)$$

where μ is the viscosity of water. The volume integral due to the additional stress can again be decomposed in two parts:

$$\int_V \sigma'_{\alpha\beta} dV = \int_{V_F} \sigma_{\alpha\beta}^F dV + \int_{V_S} \sigma_{\alpha\beta}^S dV; \quad (19.17)$$

the first one is the integral over the volume occupied by the fluid and the second one is the volume occupied by the solids.

Start with the first term on the RHS of (19.17). Let us assume that we know the flow at any point inside the volume V_F . If we did, we could calculate its derivative and symmetrize them to construct the strain matrix which we would multiply by the viscosity of water, μ to get the stress $\sigma_{\alpha\beta}^F$. Let us assume this solution is given by

$$v_\alpha = S_{\alpha\beta}^0 x_\beta + W_\alpha \quad (19.18)$$

where W_α is unknown at this moment. Remember that we have defined $\sigma_{\alpha\beta}^F$ – the contribution to the stress due to the background flow has already been taken into account in (19.16). We are now calculating the contribution to $\sigma'_{\alpha\beta}$. This contribution comes from W alone,

Then we obtain

$$\int_{V_F} \sigma_{\alpha\beta}^F dV = \mu \int_{V_F} [\partial_\alpha W_\beta + \partial_\beta W_\alpha] dV. \quad (19.19)$$

By Gauss's theorem we can convert the volume integral to a surface integral. But this surface integral will be over many disconnected surfaces. One surface is A_F the outer bounding surface of the volume V_F . The other surfaces are the inner surfaces – the surface of each of the pollens. There are N such integrals. In these surface integrals the unit outward normal to the surface appears. From the point of view of the volume V_F the unit outward normal on the surface of a pollen grain points towards the center of the grain – this is the unit inward normal from seen from the inside of the grain. This argument gives

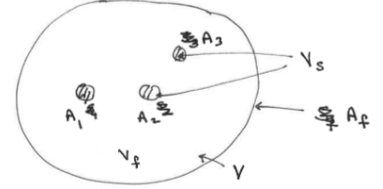


Figure 19.2: Sketch of the volume of integration in (19.15). There are N particles in volume V . The integration volume includes the fluid and also the solids. Let the volume of the k -th solid sphere be denoted by V_k and its surface as A_k . The total volume of all the solid spheres is denoted V_S and the total volume occupied by the fluid is V_F . The surface that enclosed the suspension is shown as A_F . This area is assumed to be far away from all the rigid objects hence every point on this surface is a point in a fluid. The volume V_F is like the volume inside a piece of swiss cheese – it has an external surface and many internal surfaces, one for each pollen grain.

the result:

$$\begin{aligned} \int_{V_F} \sigma_{\alpha\beta}^F dV &= -\mu \sum_k \oint_{A_k} [n_\alpha W_\beta + n_\beta W_\alpha] dA \\ &\quad + \mu \oint_{A_f} [n_\alpha W_\beta + n_\beta W_\alpha] dA \end{aligned} \quad (19.20)$$

$$= -\mu \sum_k \oint_{A_k} [n_\alpha W_\beta + n_\beta W_\alpha] dA \quad . \quad (19.21)$$

The second term on the RHS of (19.20) is zero because the integral is at a large distance away from all the pollens (at infinity) where W_α is zero for all three values of α – we have assumed that at infinity the flow is a pure strain given by $S_{\alpha\beta}^0$. We still cannot calculate the integrals in (19.20) as we do not know W_α . But the integrals are evaluated on the surface of each of the pollens. We assume that the number density of the pollens are small, so we calculate this integral on the surface of the k -th pollen we can ignore all other pollens. Then we know the solution – it is the solution for a single sphere in a flow that is purely a strain. This is given in (19.14a). So at the end we get

$$\int_{V_F} \sigma_{\alpha\beta}^F dV = -\mu N \oint_{A_k} [n_\alpha w_\beta + n_\beta w_\alpha] dA \quad . \quad (19.22)$$

Now consider the second term on the RHS of (19.17). This is an integral over the stress *inside* the rigid spheres – this is a problem in elasticity ! But we have one more trick up our sleeve. Inside a solid the internal stress satisfies the equation

$$\partial_\mu \sigma_{\alpha\mu}^S = 0. \quad (19.23)$$

This is a statement of static equilibrium inside the solid. Now we prove the following identity:

$$\frac{\partial}{\partial x_\mu} x_\beta \sigma_{\alpha\mu}^S = \frac{\partial x_\beta}{\partial x_\mu} \sigma_{\alpha\mu}^S + \partial_\mu \sigma_{\alpha\mu}^S \quad (19.24)$$

$$= \sigma_{\alpha\beta}^S \quad (19.25)$$

The partial derivative in first term on the RHS of (19.24) gives $\delta_{\mu\beta}$ and the second term is zero by virtue of (19.23). Hence we obtain

$$\int_{V_S} \sigma_{\alpha\beta}^S dV = \int_{V_S} \frac{\partial}{\partial x_\mu} x_\beta \sigma_{\alpha\mu}^S dV \quad (19.26)$$

The integral on the RHS of (19.26) is over the volume of N disconnected spheres, i.e., it is a sum over N volume integrals each over volume of one pollen grain. Each of these volume integrals can be converted to a surface integral by Gauss's theorem. There are N such surface integrals. In each of which the unit outward normal to the

surface appears. This outward normal is from the seen from the solid volume hence it is the same outward normal that has appeared in (19.22). Furthermore, as the pollen and the water is at equilibrium on the surface the fluid and solid stresses must ballance, i.e., on the surface $\sigma_{\alpha\beta}^F = \sigma_{\alpha\beta}^S$. Consequently

$$\int_{V_S} \sigma_{\alpha\beta}^S dV = \sum_k \oint_{A_k} n_\mu x_\beta \sigma_{\alpha\mu}^F \quad (19.27)$$

$$= \mu \sum_k \oint_{A_k} n_\mu x_\beta [\partial_\alpha w_\beta + \partial_\beta w_\alpha] \quad (19.28)$$

Let us now bring the whole story together. Using (19.16), (19.17), (19.22), and (19.28) we obtain

$$\mu_* S_{\alpha\beta}^0 = \mu \frac{V_F}{V} S_{\alpha\beta}^0 + \mu \frac{N}{V} \oint_{A_k} [n_\mu x_\beta (\partial_\alpha w_\beta + \partial_\beta w_\alpha) - (n_\alpha w_\beta + n_\beta w_\alpha)] \quad (19.30)$$

where the velocity field w_α is given by (19.14a) and the integral is calculated on the surface of a sphere of radius a . After substitution, integration, and minor simplifications we obtain

$$\mu S_{\alpha\beta}^0 + \frac{5}{2} \mu S_{\alpha\beta}^0 \frac{N(4/3)\pi a^3}{V} = \mu S_{\alpha\beta}^0 \quad (19.31)$$

On further simplification we obtain

$$\boxed{\frac{\mu_*}{\mu} = 1 + \frac{5}{2} \Phi} \quad (19.32)$$

where $\Phi = \frac{N(4/3)\pi a^3}{V}$ is the volume fraction occupied by the pollens.

19.4 Effective viscosity

The relation, (19.32) was first derived by Einstein (1906). In his first paper in 1906 he made a mistake in calculation which he corrected in another paper in 1911¹ from which the factor of 5/2 emerges. The method I have used was suggested by Batchelor although in his book on fluid mechanics he uses a different one where he defines the effective viscosity through rate-of-energy-dissipation. The resultant calculations are a little bit more complicated. As a mathematical curiosity the effective viscosity, not only in three-dimension, but in general d dimensions has been calculated through straightforward generalisation of the methods described here²:

$$\frac{\mu_*}{\mu} = 1 + \frac{d+2}{d} \Phi \quad (19.33)$$

The effective viscosity of suspensions is a very well-studied field within the broad field of soft-matter or chemical engineering. The

To obtain (19.31) you will need to calculate, over the solid angle Ω , the following integrals:

$$\int n_\alpha n_\beta d\Omega = \delta_{\alpha\beta} \quad (19.29a)$$

$$\int n_\alpha n_\beta n_\mu n_\nu = \quad (19.29b)$$

¹

²

Einstein relation was generalised to ellipsoids by Jefferys. It was also generalised to suspension of one liquid in another by G.I. Taylor,

$$\frac{\mu_*}{\mu} = 1 + \Phi \frac{\mu + (5/2)\mu'}{\mu + \mu'} \quad (19.34)$$

where μ' is the viscosity of the second liquid. This ignores the surface tension between the two liquids. In an ingenious experiment ³ (19.34) was used to measure the viscosity of the liquid inside the red blood cells (RBC) in the following manner. The RBC is a membrane with a liquid inside which need not necessarily have the same viscosity as the liquid outside, blood plasma. If you ignore the effects of the membrane – like ignoring surface tension – you can use Taylor's formula to calculate the effective viscosity of a suspension of RBCs. If you make a suspension of RBC in a liquids of different viscosities, i.e., different values of μ , calculate the μ_* for each of them in an experiment and then use these to calculate μ' the viscosity of the liquid inside the RBC, without ever touching the liquid !

We are still working on relaxing the two central assumptions of Einstein – dilute suspension and small Reynolds number. There is a huge body of work on this topic. In certain cases – when the Reynolds number is not small but not very large either, weakly nonlinear case – the effect of increase in Reynolds number can be modelled still as a suspension at zero Reynolds number but with an effective volume fraction that depends on the Reynolds number ⁴.

3

19.5 Discourse

ss: Is (19.17) really correct ? Did you not ignore the contribution to the stress from the surface of the pollens ?

TEACHER: Well spotted. Yes, indeed the possible contribution from surface tension is ignored.

ss: You also seem to be not very consistent about what you call the stress. Sometime you equate the stress tensor

$$\sigma_{\alpha\beta} = \mu S_{\alpha\beta} \quad (19.35)$$

where $S_{\alpha\beta}$ is the rate-of-strain matrix. But are you not forgetting the pressure ?

TEACHER: Again, you are correct. I have been somewhat cavalier with it. Equation (19.23) and its descendents are correct as they are, but (19.19) and its descendents are not really correct. Equation (19.19) should actually read

$$\int_{V_F} \sigma_{\alpha\beta}^F dV = -\delta_{\alpha\beta} \int_{V_F} p dV + \mu \int_{V_F} [\partial_\alpha W_\beta + \partial_\beta W_\alpha] dV \quad (19.36)$$

⁴F. Picano, W.-P. Breugem, D. Mitra, and L. Brandt. Shear thickening in non-Brownian suspensions: an excluded volume effect. *Phys. Rev. Lett.*, 111: 098302, November 2013

But this additional term is an isotropic contribution which does not change our conclusion. Also you do know the expression for this pressure from (19.14b).

ss: In (19.20) you ignored an integral over the surface A_f which is at large distance away from the pollens. You assumed that the velocity due to the pollens, $\mathbf{W}$ goes to zero on the surface. But the surface area increases as you go further, so although the contribution from $\mathbf{W}$ can decrease, as it is integrated over a larger area we can still get a finite contribution ?

TEACHER: Well, yes and no. It depends on how fast $\mathbf{W}$ decrease far away from the pollens. If we are far enough away we can consider all the pollens lumped together and use the solution for one pollen, $\mathbf{w}$, (19.14a).

ss: If we measure the viscosity of suspension in a very narrow channels will I still get the same result ? I can imagine a channel so small such that the rigid spheres get stuck. Clearly your theory will break down then ?

TEACHER: I am glad that you asked. Yes it will break down. Actually it will break down much earlier. If the channel width is a few integer multiple the diameter of the spheres this formula breaks down. Actually, very non-trivial behaviour can emerge ⁵.

⁵Walter Fornari, Luca Brandt, Pinaki Chaudhuri, Cyan Umbert Lopez, Dhrubaditya Mitra, and Francesco Picano. Rheology of confined non-brownian suspensions. Physical review letters, 116(1):018301, 2016

A Vectors

Exercises

1. Calculate $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$ for

$$\mathbf{A} = 2\hat{x} + 5\hat{y} + \hat{z} \quad (\text{A.1})$$

$$\mathbf{B} = -\hat{x} + 3\hat{y} - 3\hat{z} \quad (\text{A.2})$$

2. Write in index notation

(a) $\mathbf{A} \cdot \mathbf{B}$

(b) $\mathbf{A} \times \mathbf{B}$

(c) $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$.

3. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad (\text{A.3})$$

4. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 1/2 \\ 1/2 & -1 \end{pmatrix} \quad (\text{A.4})$$

5. Let ϕ be a scalar field, $\phi = 1/r$. Calculate $\nabla \phi$.

6. Let ϕ be a scalar field

$$\phi = \sin(x) + xy + \tanh(zy/x) \quad (\text{A.5})$$

Let $\mathbf{A} = \nabla \phi$. Calculate $\nabla \times \mathbf{A}$.

7. Let $\mathbf{A}$ be a vector field

$$(\text{A.6})$$

8. Calculate $\nabla \times \mathbf{A}$ for the following cases

(a) $\mathbf{A} = \sin(y)\hat{x} - \cos(z)\hat{y} + \sin(x)\hat{z}$

(b) $\mathbf{A} = \cos(x)\hat{x} + 2y\hat{y} + \sinh(z)\hat{z}$

(c) $\mathbf{A} = (1/r^2)\hat{r}$.

9. Calculate $\nabla \cdot \mathbf{A}$

(a) $\mathbf{A} = 2x\hat{x} + z\sin(y)\hat{y} - 3z\hat{z}$

(b) $\mathbf{A} = \cos(yz)\hat{x} + 2y\hat{y} + \sinh(xy)\hat{z}$

(c) $\mathbf{A} = \frac{1}{r}\hat{\theta}$

10. Let $\mathbf{A}$ be a vector field

$$\mathbf{A} = \sin(x)\hat{x} + x^2y\hat{y} + \cosh(zy/x)\hat{z} \quad (\text{A.7})$$

Let $\mathbf{B} = \nabla \times \mathbf{A}$. Calculate $\nabla \cdot \mathbf{B}$.

11. Break up the following matrix in symmetric and anti-symmetric part

$$\begin{pmatrix} 1 & 1/2 & 2 \\ 1/2 & -1 & 1 \\ 0 & -1 & 2 \end{pmatrix} \quad (\text{A.8})$$

B *Vector differential operators*

Summary of Gradient, divergence, and curl. Gauss's theorem, Stokes's theorem and Green's theorem.

Exercise 9

1. Calculate the surface integral over the unit cube

$$I = \oint \mathbf{F} \cdot \hat{\mathbf{n}} dS$$

for the vector function

$$\mathbf{F} = 2x\hat{x} + xz\hat{y} + y^2\hat{z}$$

C *Calculus of variation*

D *Summary of basic thermodynamics*

Microcanonical, Canonical, Grand Canonical, Gibbs ensemble, First and Second Law, Thermodynamic potentials, Brownian walk.

E Problems

E.1 Introduction

1. In self-avoiding random walk the mean-square-distance after N step is $\langle R^2 \rangle \sim N^{3/2}$. Calculate the pair distribution function $g(r)$.

E.2 Strain

Exercise 10

1. Visualizing strain:

(A) The following is a one-dimensional deformation field

$$u(x) = \lambda x(1 - x) \quad (\text{E.1})$$

in a medium that extends from $x = 0$ to 1. Calculate the strain. Where are the strain maximum?

(B) The following is the deformation field in two-dimensions

$$\mathbf{u} = \alpha(0, x) \quad (\text{E.2})$$

where α is small. Calculate the gradient matrix. Decompose the gradient matrix into rotation and strain. Find the two principal directions of the strain. Draw the strain ellipse. The following is the deformation field in two-dimensions

$$\mathbf{u} = \alpha(3x, x - y^2)$$

where α is small. Calculate the strain matrix for $(0,0)$ and $(1,1)$. Plot the strain ellipse for both of these cases.

(C) A displacement field is given by the following expression

$$\begin{aligned} u_x &= \alpha(5x - y + 3z) \\ u_y &= \alpha(x + 8y) \\ u_z &= \alpha(-3x + 4y + 5z) \end{aligned} \quad (\text{E.3})$$

where α is small, thereby the assumption of small displacement is satisfied. Calculate the velocity gradient matrix, decompose it into symmetric and anti-symmetric part, find out the three principal directions of strain and the equation for the strain ellipsoid in the frame of its principal axis.

2. Feelings for strain:

- (A) Under strain $s_{\alpha\beta}$ an infinitesimal volume dV changes to dV' . Show that

$$dV' = dV[1 + \text{Tr}(s_{\alpha\beta})] \quad (\text{E.4})$$

- (B) Show that a small volume change under a general deformation can be expressed as

$$\delta V = \oint_S \mathbf{u} \cdot \hat{\mathbf{n}} dS \quad (\text{E.5})$$

where $\hat{\mathbf{n}}$ is the outward normal to the surface element dS and $\mathbf{u}$ is the deformation field.

- (C) Show that under a general strain $s_{\alpha\beta}$ the angle between the x and y axis, which was initially $\pi/2$ changes by an amount $\Delta\theta = 2s_{12}$.
- (D) Given a strain matrix $s_{\alpha\beta}$ calculate the fractional change in an infinitesimal area element in the x - y plane.
- (E) Show that a diagonal element of the strain matrix gives the fractional change in length in the corresponding direction.
3. To reinforce the idea that the displacement $\mathbf{d}_2$ is indeed a rotation show that under this displacement the length of the vector $\Delta\mathbf{r}$ remains unchanged. In particular, show that the two vector $\Delta\mathbf{r}$ and $\Delta\mathbf{r}'$ have the same length.
4. A deformable material undergoes two infinitesimal deformation

$$\begin{aligned} \mathbf{r}' &= \mathbf{r} + \mathbf{u}(\mathbf{r}) \\ \mathbf{r}'' &= \mathbf{r}' + \mathbf{v}(\mathbf{r}) \end{aligned}$$

Calculate the net strain matrix. Thereby you can show how to add infinitesimal strain.

E.3 Stress

1. We set up the stress tensor from forces that acts on the faces of an infinitesimal cube. It turned out we needed 6 numbers (not nine because of symmetry of the stress tensor. Prove that these 6 (or 9) numbers are sufficient to give the internal force across

any infinitesimal surface, not necessarily the face of a cube, inside a deformed body.

2. A soft body has an external force $\mathbf{P}$ acting on its surface. The body is in mechanical equilibrium. Show that the average stress, defined by

$$\langle \sigma_{\alpha\beta} \rangle \equiv \frac{1}{V} \int_V \sigma_{\alpha\beta} dV \quad (\text{E.6})$$

is given by

$$\langle \sigma_{\alpha\beta} \rangle = \frac{1}{2V} \oint_S (P_\alpha x_\beta + P_\beta x_\alpha) dS \quad (\text{E.7})$$

3. A soft body of uniform mass density ρ in a uniform gravitational field of acceleration $\mathbf{g}$. Write the equations of mechanical equilibrium for its internal stresses.
4. **Is the stress tensor unique ?** Clearly the Newton's equations involves forces. Can there be two (or more) different stresses, i.e., two (or more) different second rank tensors that are functions of space, that gives the same internal forces (as a function of space) ? Let us define internal force per unit volume $\mathbf{f} = \mathbf{F}/\Delta V$ where $\mathbf{F}$ is given in (4.4). Then

$$f_\alpha = \partial_\beta \sigma_{\alpha\beta} \quad (\text{E.8})$$

Now consider two different stresses, $\sigma_{\alpha\beta}$ and $\tilde{\sigma}_{\alpha\beta}$ such that

$$\sigma_{\alpha\beta} - \tilde{\sigma}_{\alpha\beta} = 2\partial_\gamma \psi_{\alpha\beta\gamma} \quad (\text{E.9})$$

If both of these stresses give the same force, what kind of symmetry should $\psi_{\alpha\beta\gamma}$ have ? Do you see a similarity between this statement and gauge invariance in electrodynamics ?

E.4 Elastic energy

E.5 Elastostatics

1. Consider your space to be two-dimensional, such that

$$s_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i) \quad (\text{E.10})$$

You can now defined a two-dimensional Young's modulus

*E.6 Plates and Shells**E.7 Beams**E.8 Polymers**E.9 Hydrostatics**E.10 Dry Water**E.11 Creeping flows*

1. A thin disk is rotating with constant angular velocity Ω in the middle of a narrow cylindrical cavity filled with a fluid of dynamic viscosity μ . The width of the cavity is $2h$. Using lubrication approximation calculate the power necessary.

2.

*E.12 Navier–Stokes Equation**E.13 Boundary Layer*

1. A flat plate of area $L_x \times L_y$ is moving with a uniform velocity V along the x direction. The plate is immersed in a fluid of density ρ and kinematic viscosity ν . The fluid far away from the plate is stationary. Assume the flow is laminar.
 - (a) Calculate the net viscous force on the plate.
 - (b) Assume the plate has mass M . It has velocity V at $t = 0$ after which it is allowed to slow down, i.e., on the viscous force from the fluid acts on it. How long does it take for its velocity to become half its initial value?
2. A circular disk (radius R) is rotating in a fluid (of density ρ and kinematic viscosity ν) about its axis with constant angular velocity Ω . Far away from the disk the fluid is at rest. Assume the flow is laminar.
 - (a) Estimate the thickness of the boundary layer as a function of the radial coordinate r .
 - (b) Calculate the total viscous torque on the disk.
 - (c) Calculate the rate of energy dissipation.
 - (d) If the disk is set in motion at time $t = 0$ and then allowed to spin-down how long will it take for its angular velocity to

become half its initial value ? Assume the disk has moment of inertia I about its axis.

3. Consider the inlet of a circular pipe for the case where the Reynolds number is such that the flow is stationary but not turbulent. Typically at the inlet the flow is not parabolic. It has almost uniform velocity across the cross-section. The velocity goes to zero sharply at the boundary layer. At a large distance from the inlet a parabolic velocity profile is established. In other words, the flow profile, even at small Reynolds number, is not parabolic near the entry, this region is called the *inlet section*. The profile becomes parabolic deep inside the pipe. An estimate of the inlet section is given by distance inside the pipe at which the boundary layer fills the whole cross-section. Estimate this distance.

E.14 Waves and Instabilities

E.15 Surface Tension

E.16 Flow–structure interactions

F *Solutions*

F.1 Introduction

F.2 Strain

F.3 Stress

F.4 Elastic energy

F.5 Elastostatics

F.6 Plates and Shells

F.7 Beams

F.8 Polymers

F.9 Hydrostatics

F.10 Dry Water

F.11 Creeping flows

F.12 Navier–Stokes Equation

F.13 Boundary Layer

F.14 Waves and Instabilities

F.15 Surface Tension

F.16 Flow–structure interactions

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