Lecture 4

Two examples:

1. Sum the series (Acton page 11)

$$g(\theta) = \sum_{k=0}^{8} b_k \cos(k\theta)$$

We know b_k as fractions let us have

$$b_k = \left(\frac{1}{2}\right)^k$$

similar serieses often appear in summation of Fourier series upto a finite number of terms.

Obvious method

$$g(\theta) = b_0 + b_1 \cos\theta + b_2 \cos 2\theta + \cdots + b_8 \cos 8\theta$$

This involves 6 multiplication to produce $2\theta, \ldots 8\theta$.
7 function calls of cosines
7 multiplications again

If we want to sum the series upto n terms we need roughly speaking

$(n-2)+(n-1) = 2n-3 \simeq 2n$ multiplication

and $n-2 \simeq n$ function calls.

Each function call itself takes much longer than a flop even for something as simple as a cosine.

(Actually sines and cosines are needed so often most computers have them coded in machine language to make this faster)

However we know the recurrence relation

$$\cos(k-1)\theta - 2\cos\theta \cos k\theta + \cos(k+1)\theta = 0$$

Let us try to use this

$k=1$, $\quad 1 - 2\cos^2\theta + \cos 2\theta = 0$

$\quad \cos 2\theta = 2\cos^2\theta - 1$

$k=2 \quad \cos 3\theta = 2\cos\theta \cos 2\theta - \cos\theta$

$k=3 \quad \cos 4\theta = 2\cos\theta \cos 3\theta - \cos 2\theta$

$\vdots$

This requires just one function call (for $\cos\theta$)
Then one flop to evaluate $\cos\theta$ at each step,
total n flops
and another n flops to multiply with b_k.

But an even clever trick is to do the following

$$c_k = (2\cos\theta)\, c_{k+1} - c_{k+2} + b_k \qquad k = 8, 7, \dots 0$$

and start iterating backward from c_9 and $c_{10} = 0$

$$c_8 = (2\cos\theta)\, c_9 - c_{10} + b_8$$
$$c_7 = (2\cos\theta)\, c_8 - c_9 + b_7$$
$$c_6 = (2\cos\theta)\, c_7 - c_8 + b_6$$
$$\vdots$$

Put them in the series

$$\begin{aligned} g(\theta) = {} & c_8 \cos 8\theta \\ & + \left[c_7 - (2\cos\theta)\, c_8 + c_9\right] \cos 7\theta \\ & + \left[c_6 - (2\cos\theta)\, c_7 + c_8\right] \cos 6\theta \\ & + \dots \end{aligned}$$

$$\begin{aligned} = {} & c_8 (\cos 8\theta - 2\cos\theta \cos 7\theta + \cos 6\theta) \\ & + c_7 (\cos 7\theta - 2\cos\theta \cos 6\theta + \cos 5\theta) \\ & + \\ & + c_2 (\cos 2\theta - 2\cos^2\theta + 1) \\ & + c_1 (\cos\theta - 2\cos\theta) \\ & + c_0 \end{aligned}$$

$$= c_0 - c_1 \cos\theta$$

This involves ~~a~~ 1 function call and 8 multiplications and it is neat!

Example 2

Evaluate e^{-x} by a power series.

This series is absolutely and uniformly convergent for all x.

Let us then evaluate it for $x = 10$

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$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \ldots$$

The alternating nature coupled with the fact that at each order the magnitude of the terms increase. but the end result is improved by a very small amount means this is completely clouded by rounding error !

The solution is to actually calculate

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

and then take reciprocal. To do numerics you have to think differently.

Example 3

(Acton page 21)

Recurrence relation of common Bessel functions

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

Calculate for $x = 1$

take $J_0(1) =$

$J_1(1) =$ from tables/mathematica

and iterate forward. You shall immediately see how hopeless the situation is

Now try iterating this backword

$J_7(1) =$

$J_6(1) =$

This is better but could be improved

We know that $J_8(1) \approx 0$

Note also that for any k

$$k J_{n-1}(x) + k J_{n+1}(x) = \frac{2n}{x} k J_n(x)$$

Let us take $k J_7(1) = 1$

Then find out

$$k J_8 = 0$$

$$k J_7 = 1$$

$$k J_6 = -k J_8 + \frac{2 \cdot 6}{1} k J_7$$

$$= 12 \cdot k J_7 \quad \text{(integer multiplication)}$$

$$k J_5 = -k J_7 + 10\, k J_6$$

$$\vdots$$

Now note that

$$J_0(x) + 2\left[J_2(x) + J_4(x) + \cdots J_8(x)\right] = 1$$