Astrophysical MHD (AS7019)
Homework II

To be returned on Thursday 15th of February

All the problem carry 25 marks.

1. Solution of Stokes flow around a sphere. Continuing from where we left off in class

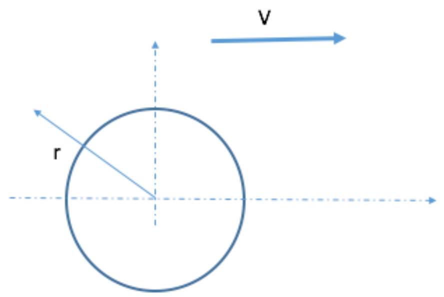

Figure 1: A solid sphere in a flow. At large distance from the sphere the flow is give by a constant velocity $\boldsymbol{V}$. The radius of the sphere is unity. The vector $\boldsymbol{r}$ is the vector to any point in space with the center of the sphere as origin.

show that

$$\chi(r) = \frac{1}{4}r^2 + Ar + \frac{B}{r} \tag{1}$$

is a solution of the biharmonic equation

$$\nabla^4 \chi = 0 \tag{2}$$

where r is the radian coordinate in spherical polar coordinate system. A and B must be chosen such that

$$u_\alpha = U_{\alpha\beta} u_\beta \tag{3}$$

with

$$U_{\alpha\beta} = \delta_{\alpha\beta} \nabla^2 \chi - \frac{\partial^2 \chi}{\partial x_\alpha \partial x_\beta} \tag{4}$$

must be zero at $r = 1$. This implies imposition of no-slip boundary conditions as all components of velocity goes to zero at the boundary of the sphere whose radius is assumed to be unity. Show that this boundary condition implies

$$\chi''(1) = \chi'(1) = 0 \tag{5}$$

From this show that $B = \frac{1}{4}$ and $A = \frac{3}{4}$. The show that for $r > 1$

$$u_\alpha = V_\alpha - \frac{3}{4}\left(\frac{V_\beta r_\beta r_\alpha}{r^3} + \frac{V_\alpha}{r} \right) - \frac{1}{4}\frac{\partial}{\partial x_\alpha}\left(\frac{V_\beta r_\beta}{r^3} \right) \tag{6}$$

2. Write down the equations of isothermal MHD and then non-dimensionalize the equations. Assume that there is a characteristic length scale $\ell = \frac{1}{k_f}$, velocity scale u, constant sound-speed c_s, a magnetic field of magnitude B_0, and a constant background density ρ_0. Show that the non-dimensionalized equations have the following dimensionless parameters:

- Reynolds number : $\mathrm{Re} = \frac{u}{\nu k_{\mathrm{f}}}$,
- magnetic Reynolds number: $\mathrm{Rm} = \frac{u}{\eta k_{\mathrm{f}}}$,
- Mach number: $\mathrm{Ma} = \frac{u}{c_{\mathrm{s}}}$,
- Alfvenic Mach number: $\mathrm{Ma_A} = \frac{u}{\mathrm{c_A}}$,

where $\mathrm{c_A} = \frac{B_0}{\rho_0 \mu_0}$.

3. Work out the solution of the solar wind problem assuming the flow is isentropic (adiabatic) instead of the isothermal assumption made in class.

4. Functional methods: Among all the curves joining two given points (x_0, y_0) and (x_1, y_1), find the one which generates the surface of minimum area when rotated about the x-axis.