Magnetohydrodynamics :

1.1 • Astrophysics :

(i) Cosmology [Gravity]

(ii) Astroparticle physics [Dark matter, Cosmic Rays, high- energy phenomenon]

(iii) Physics of plasma

+ Radiation

" Perhaps the fundamental equation that describes the swirling nebulae and the condensing, revolving and exploding stars and galaxies is just a simple equation for hydrodynamic behaviour of nearly pure hydrogen gas" Feynman , " Flow of wet water"

· plus magnetic field

Fundamental principle of astrophysics

" There are no new laws in astrophysics. It is an application of experimental laws foun terrestrially and applied astrophysically"

1.2 Plasma the 4th state of matter

It is a state of matter where there are no atoms but electrons and positive ions. Mixed up like a gas. By gas we mean that there are no order. What kind of equation will such a gas obey?

* Plasma is the most abundant state of ordinary matter.

* This is a topic of continuum mechanics, similar to fluid dynamics or elasticity.

* But plasma is more complex than ordinary gas because of it contains charges, hence can sustain magnetic field (why not electric field?)

* Fundamentally, the difficulty of dealing with plasma is the long-range nature of the Coulomb interaction, however 'shielding' provides some help. We shall come back to this topic later.

* Fusion plasma and the solution to all our problems.

1.3 Continuum mechanics:

- Traditionally derived as many-body formulation of Newton's laws. But with additional constitutive ~~equ~~. coefficients (elastic coefficients, viscosity, thermal conductivity)

- Theoretical physics and length and time scales. The concept of "effective theories."

~~Quantum mechanics~~ → ~~many-body con~~

"fundamental theories"

↓

quantum mechanics → few particles q. mech. quantum chaos

↓

classical mechanics

↓

few particles classical mech. ↔ chaos theory stat. mech. | many-body class. mech. | many body ~~stat~~ q mech. (superconductivity. ...)

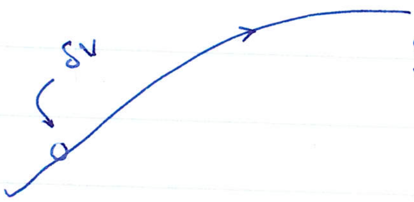
continuum mechanics.
can also be formulated as
non-equilibrium statistical mechanics.

× Each step is a change of scale, "coarse graining"

~~× let us start by again~~

× Equations of a simple fluid :

× Apply Newton's laws to a fluid element :



$\rho\,\delta V$ (acceleration)

= pressure force

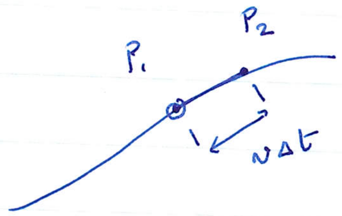
+ body forces (e.g. gravity)

+ viscous forces.

body forces : $-\rho \nabla \phi$

pressure forces : $-\nabla p$

<u>acceleration</u> : $\frac{d\vec{v}}{dt}$



$$\Delta v = v(x + \Delta x, \ y + \Delta y, \ z + \Delta z, \ t + \Delta t)$$

$$= v(x + v_x \Delta t, \ y + v_y \Delta t, \ z + v_z \Delta t, \ t + \Delta t)$$

$$= v(x, y, z) + \frac{\partial v_x}{\partial x} v_x \Delta t + \cdots + \frac{\partial v}{\partial t} \Delta t + \text{h.o.t}$$

The acceleration

$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = (v \cdot \nabla) v + \frac{\partial v}{\partial t}$$

Putting togethere :

$$\rho \left[\frac{\partial v}{\partial t} + (v \cdot \nabla) v \right] = -\nabla p - \rho \nabla \phi + \text{viscous force.}$$

Beginning of hydrodynamics.

× A second way to derive hydrodynamics:

- look for conserved quantities:

mass, momentum, energy.

Each conserved quantity will have a density and a current.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot g = 0$$

$$\frac{\partial g_i}{\partial t} + \nabla \cdot \Pi_{ij} = 0$$

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot j_\varepsilon = 0$$

- clearly current of mass is the momentum.

$$g = \rho v$$

~~This is the~~ current of a conserved quantity is also conserved.

To proceed let us be a little bit more careful.

- We consider a fluid that has local thermodynamic equilibrium. In other words one can define a local temperature.
- So we are dealing with thermodynamics of moving systems.
- Remind ourselves some thermodynamics:

- An interacting classical system is described by a Hamiltonian $\mathcal{H}$

All of thermodynamics is in the partition function

$$Z = \mathrm{Tr}\, e^{-\beta \mathcal{H}}$$

integral over all degrees of freedoms.

$$\mathcal{F} = -T \ln Z \qquad \text{Helmholtz potential.}$$

$$= E - TS.$$

~~But now we are in a moving system.~~

$$d\mathcal{F} = dE - TdS - SdT \qquad TdS = dE + pdV$$

$$= -SdT - pdV$$

Now consider ~~a~~ thermodynamics with motion.

$$Z_N(T, V, v) = e^{\beta N m v^2/2} Z_N(T, V, 0)$$

Because in a classical system velocity is independent of position.

$$\Rightarrow \quad \mathcal{F}(T, V, N, v) = \mathcal{F}(T, V, N, 0) - \frac{1}{2} N m v^2$$

The momentum operator

$$P_j = - \left. \frac{\partial \mathcal{F}}{\partial v_j} \right|_{T, V, N}$$

$$= N m v_j$$

$\Rightarrow$ ~~$d\mathcal{F}$~~ $=$

$$\Rightarrow \quad d\mathcal{F} = - S\,dT - p\,dV - P \cdot dv$$

Now introduce the grand potential:

$$\mathcal{A} = \mathcal{F}_0 - \mu N$$

chemical potential.

Clearly

$$\cancel{\mu} \quad \cancel{\mu_0}$$

$$\mu = \frac{\partial F}{\partial N} = \mu_0 - \frac{1}{2} m v^2$$

The grand potential is related to the pressure by

$$\Omega = - V p(\mu, T\, v)$$

$$\Rightarrow \quad p = - \frac{\Omega}{V}$$

$$= -\frac{1}{V} \left(\mathcal{F} - \mu N \right)$$

$$= - \frac{1}{V} \left[E - TS - \mu N - \frac{1}{2} N m v^2 \right]$$

$$= - \varepsilon - Ts - \alpha \rho - \vec{g} \cdot \vec{v}$$

$$\rho = \frac{Nm}{V}, \quad \alpha = \frac{\mu}{m},$$

Then the entropy eqn.

$$T ds = d\varepsilon - \alpha\, d\rho - \vec{v} \cdot d\vec{g}$$

Now write an equation of entropy transport:

$$T\left[\frac{\partial s}{\partial t} + \nabla\cdot\left(v s + \frac{\Phi}{T}\right)\right]$$

$$= -\varphi\cdot\frac{\nabla T}{T} - (g - \rho v)\cdot\nabla\alpha$$

$$-\left(\Pi_{ij} - p\,\delta_{ij} - v_i g_j\right)\nabla_i v_j$$

Demand zero dissipation

$$g = \rho\,\vec{v}$$

$$\Pi_{ij} = p\,\delta_{ij} + v_j g_i$$

$$j_\varepsilon = (\varepsilon + p)\,v = \left(\varepsilon_0 + p + \tfrac{1}{2}\rho v^2\right)v$$

$$\boxed{\begin{aligned} &\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v) = 0 \\ &\partial_t(\rho v) + \nabla\cdot(\rho v v) = -\nabla p \\ &\partial_t s + \nabla\cdot(v s) = 0. \end{aligned}}$$

Dissipationless hydrodynamics.

In this lecture we shall again derive the equations of MHD ~~but~~ but this time with slightly more care.
We start again by writing down equations for conserved quantities. [per unit volume; or density variables]

mass $\partial_t \rho + \mathrm{div}\, g = 0$ 1a

momentum $\partial_t g + \mathrm{div}\, \pi_{ij} = 0$ 1b

energy $\partial_t \varepsilon + \mathrm{div}\, j_\varepsilon = 0$ 1c

Without an expression for the ~~the~~ currents these equations are ~~as~~ useless.
The current for mass density is clearly

$$g = \rho v \quad \text{momentum.} \qquad 2a$$

The current for momentum density is clearly

$$\Pi_{ij} = \rho v_i v_j + \text{other forces.}$$

Let us assume that there are no external forces. Then the only force is pressure.

$$\Pi_{ij} = \rho v_i v_j + p\,\delta_{ij} \qquad 2b.$$

The derivation of the ~~heat~~ energy current is more involved. We need to remember thermodynamics.

The second law of thermodynamics for a fluid system with fixed mass is

pressure

$$T\,dS = dE + p\,dV \qquad (3)$$

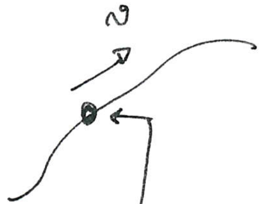
temperature entropy internal energy

Let us define the extensive quantities a per unit mass:

$$T\,d\tilde{s} = d\tilde{e} + p\,d\left(\frac{1}{\rho}\right)$$

$$= d\tilde{e} - \frac{p}{\rho^2}\,d\rho$$

$$\Rightarrow \quad \rho T\,d\tilde{s} = \rho\,d\tilde{e} - \frac{p}{\rho}\,d\rho$$



Now apply this equation to a "fluid parcel"
Here $\tilde{e}$ is the internal energy per unit mass, it does not contain the kinetic energy of the fluid parcel.

$$\rho T D_t \tilde{s} = \rho D_t \tilde{e} - \frac{p}{\rho} D_t \rho \qquad (4)$$

Now use 1a and 2a to write the continuity eqn.

$$\partial_t \rho + \operatorname{div}(\rho v) = 0$$

$$\Rightarrow \quad D_t \rho + \rho \operatorname{div} v = 0 \qquad (5)$$

Substitute (5) in (4) to obtain:

$$\rho T D_t \tilde{s} = \rho D_t \tilde{e} + p \operatorname{div} v \qquad - (6).$$

Consider dissipation less hydrodynamics:

$$D_t \tilde{s} = 0 \qquad - (7)$$

Now note the following identity

$$\rho D_t \psi = \partial_t(\rho\psi) + \operatorname{div}(v\psi) \qquad - (8)$$

true for any ~~quantity~~ ~~ψ~~ density variable ψ when ρ and v together satisfies the continuity eqn. The equations for dissipationless hydrodynamics is then:

$$\partial_t \rho + \operatorname{div}(\rho v) = 0 \qquad 9a$$

$$\partial_t(\rho v) + \operatorname{div}\left(p\delta_{ij} + \rho v_i v_j\right) = 0 \qquad 9b$$

$$\partial_t s + \operatorname{div}(v s) = 0 \qquad 9c$$

where $s = \rho\tilde{s} \equiv$ entropy per unit volume.
The last equation is obtained by ~~to~~ using (7) and (8)

2.2

An alternative formulation of the problem uses the energy equation and its current.
(7) implies:

$$\rho D_t \tilde{e} + p \operatorname{div} v = 0$$

$$\Rightarrow \quad \partial_t(\rho e) + \operatorname{div}(\rho e v) + p \operatorname{div} v = 0 \qquad (10)$$

~~Remember~~, ~~the~~ Here $e = \rho\tilde{e} \equiv$ internal energy per unit volume.

The total energy per unit volume:

$$\mathcal{E} = e + \underbrace{\tfrac{1}{2}\rho v^2}_{\text{kinetic energy.}}$$

From (9b)

$$\partial_t(\rho v_i) + \partial_j(\rho v_i v_j) + \partial_i p = 0$$

multiply by v_i and sum over i to obtain

$$\partial_t\left(\frac{\rho v_i^2}{2}\right) + \partial_j\left(v \rho \frac{v^2}{2}\right) + v\cdot\nabla p = 0 \qquad -(11)$$

Add (10) and (11) to obtain

$$\partial_t \mathcal{E} + \operatorname{div}\left[v(\mathcal{E}+p)\right] = 0 \qquad -(12)$$

$\Rightarrow$ The heat flux: $\quad j_{\mathcal{E}} = v(\mathcal{E}+p) \qquad$ 13a

with $\quad \mathcal{E} = e + \frac{1}{2}\rho v^2 \qquad$ 13b

2.3 Dissipation:

Once we allow for dissipation the entropy equation will change to:

$$\partial_t s + \operatorname{div}\left(v s + \frac{Q}{T}\right) = 0 \qquad - 14.$$

where Q is the heat flux

There can be several contributions to this heat flux.

i)

(i) heat transport due to gradient of temperature

$$\varphi = - K \nabla T$$

K — thermal conductivity

(a) K must be a scalar for an isotropic fluid
(b) K must be positive for entropy to ~~always~~ to be non-decreasing.

(ii) heat transport due to viscous heating.

In a fluid there can be momentum transport because of viscous stresses.

$$\Pi_{ij} = \rho v_i v_j + p \delta_{ij} + \sigma_{ij}$$

Then $\quad Q = - v_j \sigma_{ij}$

In general $\quad \sigma_{ij} = \eta_{ijkl} \partial_k v_l$

η_{ijkl} — viscosity tensor

For an isotropic fluid there are only two independent contribution to a 4th rank tensor

$$\sigma_{ij} = \eta \left(\partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \partial_k v_k \right) + \zeta \delta_{ij} \partial_k v_k$$

η — shear viscosity

ζ — bulk viscosity.

The equations of viscous hydrodynamics :

$$\partial_t \rho + \operatorname{div}(\rho v) = 0$$

$$\partial_t(\rho v) + \operatorname{div}\left(p\,\delta_{ij} + \rho\, v_i v_j - \sigma_{ij}\right) = 0$$

$$\sigma_{ij} = \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\,\delta_{ij}\,\partial_k v_k\right) + \zeta\,\delta_{ij}\,\partial_k v_k$$

$$\partial_t s + \operatorname{div}\left(v s + \frac{q}{T}\right) = 0$$

$$q = -\kappa \nabla T - v_j \sigma_{ij} + \text{radiation}$$

× In addition we need an equation of state often the ideal gas equation

$$p = \rho T$$

× Incompressible approximation :

$$\rho : \text{constant} \equiv 1.$$

$$\Rightarrow \quad \operatorname{div} v = 0$$

$$\partial_t v + \operatorname{div}(v_i v_j) = -\nabla p + \nu \nabla^2 v \qquad \nu = \frac{\eta}{\rho}$$

– The Navier-Stokes equation.

× Isothermal approximation.

3.1 How to include the magnetic field.

* We consider a plasma that is a very good conductor. The charge separation is negligible. Electrostatic field is almost zero.

The force on a current density J is

$$J \times B.$$

Hence the contribution of magnetic field to the momentum eqn. is

$$\partial_t (\rho v) + \mathrm{div}\left(p\delta_{ij} + \rho v_i v_j - \sigma_{ij}\right) = J \times B.$$

Maxwell's eqn

$$\nabla \times B = \mu_0 J - \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

Assume that all time dependence is much slower compared to speed of light; hence ignore the displacement current.

$$\Rightarrow \quad \partial_t (\rho v) + \mathrm{div}\left(p\delta_{ij} + \rho v_i v_j - \sigma_{ij}\right) = \frac{1}{\mu_0}(\nabla \times B) \times B$$

Using vector identities one can write this as

$$\partial_t (\rho v) + \mathrm{div}\left[p\delta_{ij} + \rho v_i v_j - \sigma_{ij} - B_i B_j + \delta_{ij}\frac{B^2}{2}\right] = 0$$

- Magnetic field contributes to pressure:

$$\delta_{ij}\left(p + \frac{B^2}{2}\right)$$

- Maxwell's stress: $B_i B_j$

The eqn describing the evolution of magnetic field

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \text{Faraday's law.}$$

Ohm's law

$$J = \sigma\left(E + v \times B\right)$$

$\cancel{\sigma}$: ~~thermal~~ ~~conductiv~~

σ: electrical conductivity.

$$\Rightarrow \quad \frac{\partial B}{\partial t} = \nabla \times \left[v \times B - \frac{1}{\sigma} J\right]$$

$$= \nabla \times \left(v \times B - \frac{1}{\mu_0 \sigma} \nabla \times B\right)$$

$$= \nabla \times (v \times B) + \eta \nabla^2 B \qquad \eta = \frac{1}{\mu_0 \sigma}$$

$\downarrow$ magnetic diffusivity

The magnetic field would also contribute to energy:

$$\mathcal{E} = e + \frac{1}{2}\rho v^2 + B^2$$

The energy eqn.

$$\partial_t \mathcal{E} + \operatorname{div} j_{\mathcal{E}} = 0$$

The magnetic contribution to the energy flux must be the Poynting flux

$$
\begin{aligned}
S &= E \times B \\
&= \left(-v \times B + \frac{1}{\sigma} J\right) \times B \\
&= B \times (v \times B) + \frac{1}{\sigma} J \times B
\end{aligned}
$$

$$
j_\varepsilon = \rho v \left(e + \tfrac{1}{2}\rho v^2 + p\right) + B \times (v \times B) + \eta (\nabla \times B) \times B - v\sigma_{ij} - K \nabla T
$$

The ~~contribu~~ magnetic contribution to the entropy eqn. must be the Joule heating

$$
\partial_t s + \mathrm{div}\left(v s + \frac{q}{T}\right) = 0
$$

where

$$
q = -K \nabla T - v \sigma_{ij} + \eta (\nabla \times B) \times B
$$

- Incompressible MHD equations
- Isothermal MHD equations.

The magnetic part always includes the constraint

$$
\nabla \cdot B = 0
$$

This can be always satisfied by solving for the vector potential instead of B

$$
B = \nabla \times A
$$

The evolution equation for the vector potential is :

$$\partial_t A = v \times B - \frac{1}{\sigma} J$$

× conservation laws :

The MHD equations have two conserved quantities in the ideal (dissipation less case)

1. The total energy $\mathcal{E} = e + \frac{1}{2}\rho v^2 + B^2$ integrated over all volume:

$$E = \int_V \mathcal{E}(\vec{x})\, dV = \text{constant}$$

2. The magnetic helicity

$$\mathcal{H} = \int_V \vec{A} \cdot \vec{B}\, dV$$

$$\partial_t \mathcal{H} = \int \left[\partial_t A \cdot B + A \cdot \partial_t B \right] dV$$

$$\int \partial_t A \cdot B\, dV = \int (v \times B) \cdot B\, dV = 0$$

$$\int A \cdot \partial_t B\, dV = \int A \cdot \nabla \times (v \times B)\, dV$$

$$\nabla \cdot (P \times Q) = P \cdot (\nabla \times Q) + Q \cdot (\nabla \times P)$$

$$\vec{A} \cdot \nabla \times (v \times B) = \nabla \cdot (A \times v \times B) - (v \times B) \cdot \nabla \times A$$

$$= \nabla \cdot (A \times v \times B) - (v \times B) \cdot B$$

$$\Rightarrow \quad \partial_t \mathcal{H} = \int \operatorname{div}\left(\dot{A} \times \nabla \times B\right) dV$$

$$= \oint_S \left(A \times \nabla \times B\right) \cdot \hat{n} \, ds$$

Assuming velocities, and magnetic field has zero normal component at surfaces at ~~infinit~~ at the boundary we have

$$\partial_t \mathcal{H} = 0$$

$\Rightarrow$ $\mathcal{H}$ is a conserved quantity.

Also, when $\mathcal{H}$ is conserved it is gauge independent.

Exercise Set 1.

To be returned on 28 Jan 2016

1. Show that if ρ and v satisfies the continuity eqn. then for any density variable ψ the following identity holds:

$$\rho D_t \psi \equiv \rho(\partial_t + v.\nabla)\psi$$ (5 marks)

$$= \partial_t(\rho\psi) + \operatorname{div}(v\rho\psi)$$

2. Prove the vector identity

$$B\times(\nabla\times B) = \frac{1}{2}\nabla B^2 - (B\cdot\nabla)B$$ (5 marks)

do you need to use $\operatorname{div} B = 0$. ?

3. Argue why ~~there are~~ only two independent numbers are necessary to describe an isotropic tensor of rank 4. You can look up the argument in a book if you do not remember it. (3 marks)

4. Show that the total energy

$$\mathcal{E} \equiv \int_V \left[\frac{1}{2}\rho v^2 + e + B^2\right] dV$$

e: internal energy per unit volume.

is a conserved quantity of the ideal MHD equations. Here the integral is over ~~all space~~. a periodic domain. (7 marks)

Lecture 3

- Incompressible hydrodynamics.

The equations of dissipative hydrodynamis that we wrote down are:

continuity eqn $$\partial_t \rho + \mathrm{div}(\rho v) = 0$$

momentum eqn $$\partial_t(\rho v) + \mathrm{div}\left(\rho v_i v_j + p\delta_{ij} + \sigma_{ij}\right) = 0$$

with $$\sigma_{ij} = \eta\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_k v_k\right) + \zeta\,\delta_{ij}\partial_k v_k$$

~~energy~~
entropy eqn

$$\partial_t s + \mathrm{div}\left(v s + \frac{q}{T}\right) = 0$$

with $$q = -\kappa\nabla T - v_j\sigma_{ij}$$

with an equation of state

This makes a complete dynamical theory. The picture becomes a lot simpler if we consider incompressible fluids.

The incompressible approximation:

$$\rho = \text{constant} = 1$$

$$\Rightarrow \quad \nabla \cdot v = 0$$

The dynamical theory ~~dets~~ gets the following simplified form:

$$\partial_t (\rho v) + \operatorname{div} (\rho v_i v_j + p\, \delta_{ij} \mp \sigma_{ij}) = 0$$

with $\quad \sigma_{ij} = \eta \left(\partial_i v_j + \partial_j v_i \right)$

$$\Rightarrow \quad \rho \left[\partial_t v + (v \cdot \nabla) v \right] = - \nabla p + \eta \operatorname{div} \left(\partial_i v_j + \partial_j v_i \right)$$

$$\Rightarrow \quad D_t v = - \frac{\nabla p}{\rho} + \frac{\eta}{\rho} \nabla^2 v$$

$\frac{\eta}{\rho} \equiv \nu$ is the kinematic viscosity.

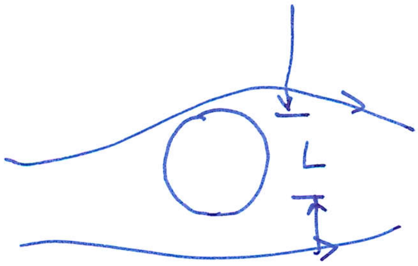
We obtain the famous Navier-Stokes equation

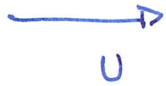
$$\boxed{\begin{aligned} \nabla \cdot v &= 0 \\ \partial_t v + (v \cdot \nabla) v &= - \nabla p + \nu \nabla^2 v \end{aligned}}$$

This makes a complete dynamical theory

3.2 Reynold's number :

consider flow of ~~in~~ incompressible fluid around a sphere.





The boundary conditions are that the velocity is a constant at infinity, and both tangential and vertical component of velocity are zero at the surface of the sphere.

Let us try to non-dimensionalize the governing equations.

velocity : U

length : L

time : $\frac{L}{U}$

$u \longrightarrow U\tilde{u}$

non-dimensional velocity.

How should we non-dimensionalize pressure?

~~∇p has the same dimensi~~

Better take a curl of the governing equation

$$\cancel{\Omega} = \cancel{\nabla \times v} \qquad \omega = \nabla \times v$$

$$\partial_t \omega + \nabla \times \left[(v \cdot \nabla) v \right] = 0 + \nu \nabla^2 \omega$$

Using vector identities you can show that

$$\cancel{\nabla \times} (v \cdot \nabla) v = \cancel{\Omega \times v} \; \omega \times v$$

Hence:

$$\partial_t \omega + \nabla \times (\omega \times v) = \nu \nabla^2 \omega$$

Now, non-dimensionaliz:

$$[\omega] = \frac{1}{T}$$

$$[\partial_t \omega] = \frac{1}{T^2}$$

$$\frac{1}{T^2} \tilde{\partial}_t \tilde{\omega} + \cancel{\frac{}{L}}$$

$$\frac{1}{T}\frac{1}{T}\,\partial_t \omega + \frac{1}{L}\,\frac{1}{T}\,\frac{L}{T}\,\nabla\times(\omega\times v)$$

$$= \nu\,\frac{1}{L^2}\,\frac{1}{T}\,\nabla^2\omega$$

$$\Rightarrow \quad \partial_t \omega + \nabla\times(\omega\times v)$$

$$= \frac{\nu T}{L^2}\,\nabla^2\omega$$

$$= \frac{\nu}{L^2}\,\frac{L}{U}\,\nabla^2\omega = \frac{\nu}{UL}\,\nabla^2\omega.$$

$$\Rightarrow \quad \boxed{\partial_t \omega + \nabla\times(\omega\times v) = \frac{1}{Re}\,\nabla^2\omega}$$

The whole problem reduces to a single dimensionless parameter, the Reynold's no.

* The limit of $Re \to 0$ is not the same as $Re = 0$ because this is a problem of singular perturbation theory. In other words $Re \to \infty$, which implies $\nu \to 0$ is ~~not~~ the problem does not reduce to ideal problem.

The vorticity eqn :

- $$\partial_t \omega + \nabla \times (\omega \times v) = \frac{1}{Re} \nabla^2 \omega$$

$$\Rightarrow \quad \partial_t \omega = \nabla \times (v \times \omega) + \frac{1}{Re} \nabla^2 \omega$$

Is exactly the same as the eqn. for the magnetic field

$$\partial_t B = \nabla \times (v \times B) + \frac{1}{Re_M} \nabla^2 B .$$

where $Re_M \equiv$ magnetic Reynold's number

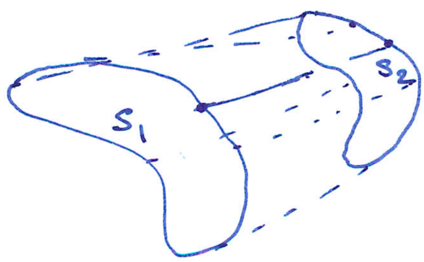
$$= \frac{UL}{\eta}$$

- In the ideal case, $\nu = 0$

$$\partial_t \omega + \nabla \times (\omega \times v) = 0$$

Kelvin's vorticity theorem :

Consider a open surface in a fluid, ~~that~~ The surface is made out of fluid parcels.



With time the parcels will move.

The flux of vorticity through this surface will remain a constant in time

$$\int_{S_1} \omega . \hat{n} \, ds = \int_{S_2} \omega . \hat{n} \, ds$$

or

$$D_t \int_S \omega . \hat{n} \, ds = 0$$

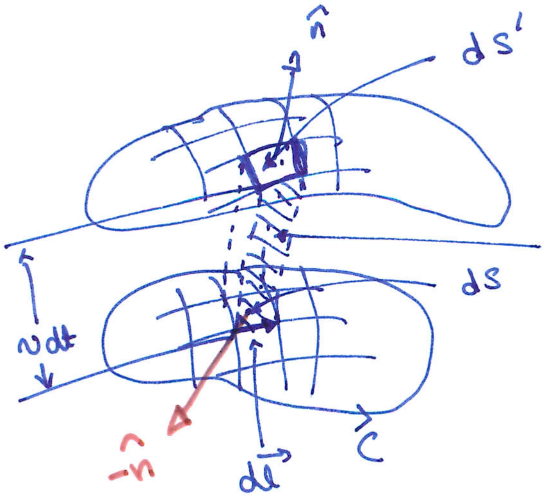
<u>proof</u>

$$D_t \int_S \omega . \hat{n} \, ds$$

$$= \int \frac{\partial \omega}{\partial t} . \hat{n} \, ds + \cancel{\omega . d} + \int \omega . D_t (\hat{n} \, ds)$$

$$= - \int \nabla \times (\omega \times v) . \hat{n} \, ds + \text{2nd term}$$

$$= - \oint_C (\omega \times v) . dl + (\text{2nd term})$$



$\cancel{d}\ \vec{dl} \times \vec{v}\,dt$

clearly :

$$\hat{n}'ds' - \hat{n}ds + \begin{pmatrix}\text{vertical}\\ \text{area}\end{pmatrix} = 0$$

$$\omega \cdot \frac{d}{dt}(\hat{n}\,ds) \qquad \hat{n}'ds' - ds\,\hat{n} - dt\oint \vec{v} \times \vec{dl} = 0$$

$$D_t(\hat{n}\,ds) = \lim_{dt \to 0} \frac{\hat{n}'ds' - \hat{n}\,ds}{dt}$$

$$= \oint \vec{v} \times \vec{dl}$$

$$\int \omega \cdot D_t(\hat{n}\,ds) = \int \oint \omega \cdot (\vec{v} \times \vec{dl})$$

$$= \int \oint (\omega \times v) \cdot dl$$

$$= \oint_C (\omega \times v) \cdot dl$$

(Because all the inner contributions cancel out)

$\cancel{AD}$ $\qquad \cancel{D_t \oint \omega \cdot \hat{n}}\, ds$

$$\Rightarrow \quad D_t \oint \omega \cdot \hat{n}\, ds = 0$$

In other words

$$\int_S \omega \cdot \hat{n}\, ds = \int_S (\nabla \times v) \cdot \hat{n}\, ds$$

$$= \oint_C v \cdot dl \equiv \text{circulation}$$

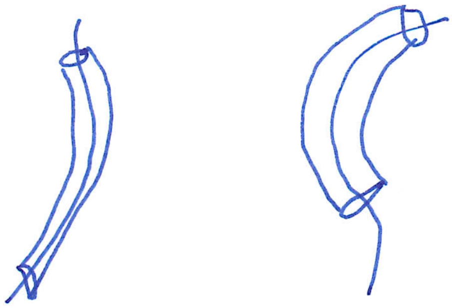
Circulation is a conserved quantity in ideal hydrodynamics.

* The same equation is obeyed by the magnetic field hence in ideal MHD the ~~*~~ flux of a magnetic field through a surface is conserved. This is called the theorem of "flux freezing".

(Alfven ~~192~~ 1942)

× Possible consequences of flux freezing.

The magnetic field is slaved to the flow. If you know how the flow goes ~~you~~ you can predict the field.



Two fluid parcels on the same field line remains on it.

* If a structure collapses under gravity its magnetic field can become very intense.

The vorticity eqn:

$$\partial_t \omega + \nabla \times (\omega \times v) = \nu \nabla^2 \omega$$

$$\nabla \cdot v = 0$$

$$\omega = \nabla \times v$$

makes a complete dynamical theory.

* We obtain v by the Biot-Savant law.

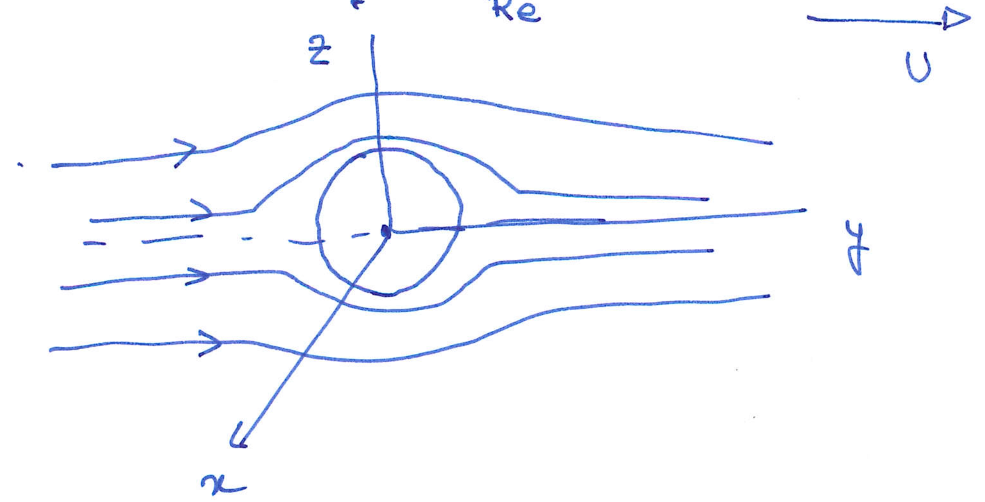
* In the non-ideal case ~~to~~ vorticity diffuses.

Some solutions of the Navier-Stokes eqn.

1. Stokes eqn.

$$\nabla \cdot v = 0$$

$$\partial_t v = \frac{1}{Re} \nabla^2 v - \nabla p$$



Look for stationary solutions:

$$\nabla \cdot v = 0$$

$$\frac{1}{Re} \nabla^2 v - \nabla p = 0$$

Look for solutions in the following form:

$$v_i = U_{ij} a_j \,, \qquad p = \Pi_j a_j$$

where $\vec{a}$ is a constant vector.

$$U_{ij} = \delta_{ij} \nabla^2 \chi - \frac{\partial^2}{\partial x_i \partial x_j} \chi$$

$$\cancel{\nabla \cdot v} = \cancel{\frac{\partial}{\partial x_j}} \quad \cancel{\delta_{ij} \nabla^2 x} \; a_i$$

$$\nabla \cdot v = \frac{\partial}{\partial x_i} \left[\delta_{ij} \nabla^2 \chi \, a_j - \frac{\partial^2 \chi}{\partial x_i \partial x_j} a_j \right]$$

$$= \left[\delta_{ij} \nabla^2 \frac{\partial \chi}{\partial x_i} a_j - \frac{\partial}{\partial x_j} \nabla^2 \chi \, a_j \right] = 0$$

$$\nabla^2 v = \left[\delta_{ij} \nabla^4 \chi - \frac{\partial^2}{\partial x_i \partial x_j} \nabla^2 \chi \right] a_j$$

Choose $\quad \Pi_j = \dfrac{\partial}{\partial x_j} \nabla^2$

Then we obtain

$$\boxed{\nabla^4 \chi = 0}$$

a biharmonic eqn.

Lecture 4

Solutions of equations of MHD (contd.)

- comments of the Stokes solution.

To solve $\nabla\cdot v = 0$, $\nu\nabla^2 v - \nabla p = 0$ } we used a prescription of $v_i = U_{ij} a_j$, $p = \Pi_j a_j$

with $U_{ij} = \delta_{ij}\nabla^2\chi - \dfrac{\partial^2\chi}{\partial x_i \partial x_j}$

with χ ultimately obtained by solving ~~dx~~ $\boxed{\nabla^4\chi = 0}$

where did this choice come from?

~~If we had consider~~

The idea is to solve the original pde we want to turn it into a pde which has less variables. The original PDE was for three components of velocity with one constraint. ~~No~~ If we ~~are~~ were in ~~three dimensio~~ two dimensions we could write: $\nu\nabla^2(\nabla\times v) = 0 \Rightarrow \nabla^2\omega = 0$ [which would still be one PDE with two variables] using $\omega = \nabla^2\psi$ where ψ is the streamfunction we could write $\nabla^4\psi = 0$. But in 3-d this would not ~~hold~~ ~~as~~ ~~ψ~~ ~~should~~ help; because ψ_i would also be a vector function. But the idea is somewhat similar. We write ~~$v \sim \nabla\times\nabla\times\psi$~~ ~~$v \sim \nabla\times\psi$~~

~~and~~ ~~$\omega =$~~ $v \sim \nabla\times\nabla\times\psi, \sim \nabla\nabla\psi - \nabla^2\psi$

which is roughly what U_{ij} is such that $\nabla\cdot v = 0$

Force - free Solution

4.1. ~~Incompressible~~ Isothermal, ideal MHD eqn and some of it's solutions :

consider :

$$\partial_t \rho + \operatorname{div}(\rho v) = 0$$

$$\partial_t (\rho v) + \operatorname{div}(\rho v_i v_j + p\delta_{ij}) = J\times B$$

$$\partial_t B = \nabla\times(v\times B) \qquad c_s^2 = \frac{\gamma p}{\rho} = \text{constant}$$

Eqns. of isothermal MHD.

Clearly : $v = 0$ is a solution ~~even~~ (hydrostatics) which ~~also~~ would still remain a solution if we consider viscosity. If we have

$$\boxed{J\times B = 0}$$ force-free magnetic field.

One solution is $B =$ constant. Which is also a solution if we include η. But there are other solutions

If $\underbrace{\nabla\times B = \Lambda B}_{B \text{ is an eigenfunction of the curl operator.}}$, $\Rightarrow J\times B = 0$.

In ~~an~~ Cartesian coordinates, the solutions are called Beltrami solutions. Remember that in the incompressible case the flow eqn can be written as $\partial_t v + \omega\times v = \nu\nabla^2 v - \nabla p$

if $\omega = \Lambda v$, then the non-linear term is zero!

so if v and B are both Beltrami, $J\times B = 0$, $\omega\times v = 0$

i.e. ~~both~~ both the nonlinear terms are zero.

Taking another curl : $\nabla\times\nabla\times B = -\nabla^2 B = \Lambda^2 B$

$$\Rightarrow \eta\nabla^2 B = -\Lambda^2 B$$

Also : $$\boxed{\nabla^2 B + \Lambda^2 B = 0}$$ vector Helmholtz Eqn.

The Beltrami fields in Cartesian:

$$u_x = A \sin \Lambda z + C \cos \Lambda z$$

$$u_y = B \sin \Lambda x + A \cos \Lambda z$$

$$u_z = C \sin \Lambda y + B \cos \Lambda x$$

– Arnold - Beltrami - Childress is a solution of the incompressible Navier - Stokes equations.

$$\partial_t u + \cancel{\ } \omega \times u = \nu \nabla^2 u - \nabla p$$

$$\nabla \cdot u = 0$$

The solution in spherical polar coordinates is called Chandrasekhar - Kendall functions

To solve the vector Helmholtz eqn, write down first the solutions to the scalar Helmholtz eqn.

$$\nabla^2 \psi + \Lambda^2 \psi = 0$$

Then define: $B = T + S$

with $T = \nabla \times (\hat{e} \psi)$ and $S = \frac{1}{\Lambda} \nabla \times T$

any constant unit vector

Also note that the force-free solutions are helical; depending on what you choose as Λ.

$$\nabla \times B = \Lambda B \quad \Rightarrow \quad \cancel{\text{if}} \quad \cancel{A =} \ \cancel{\nabla \times B};$$

$$\cancel{\nabla \times A} = \cancel{\nabla \times} \ \cancel{\nabla \times}$$

$$\Rightarrow \text{ if } \quad B = \nabla \times A, \quad A = \frac{1}{\Lambda} B$$

such that $\nabla \times A = \frac{1}{\Lambda} \nabla \times B = \frac{1}{\Lambda} \Lambda \cancel{\nabla \times} B = \cancel{\ } B$

<u>Helicity</u>: $\mathcal{H} = \int \vec{A} \cdot \vec{B} \, dV = \frac{1}{\Lambda} \int \vec{B} \cdot \vec{B} \, dV = \frac{1}{\Lambda} E_M$

magnetic energy; always +ve.

~~4.2 Pressure balanced plasma column~~ :

4.2 Taylor's theory of decay in plasma :

How should plasma decay from an initial arbitrary configuration ?

Taylor's hypothesis : In the limit of high Rem or small η plasma shall decay to a state with minimum energy but constant helicity.

[What ? Is not that strange ? One conserved quantity gets minimized and the other one remains constant ?]

Let us see what the consequence will be.

Energy $\equiv E_M = \int \frac{B^2}{2} dV$ Helicity $\equiv \mathcal{H} = \int A \cdot B \, dV$

Then the plasma will decay to a state given by :

$$\delta \left[E_M + \underbrace{\Lambda}_{\text{Lagrange's multiplier.}} \mathcal{H} \right] = 0$$

$$\Rightarrow \delta \int \left[\frac{B^2}{2} + (A \cdot B) \Lambda \right] dV = 0$$

$$\Rightarrow \int \left[B \cdot \delta B + \Lambda \left(B \cdot \delta A + A \cdot \delta B \right) \right] dV = 0$$

~~consider : $\int (B \cdot \delta A) \, dV = \int \delta A \cdot (\nabla \times A) \, dV$~~

~~$= \nabla \cdot (\delta A \times A) + \delta A \cdot (\nabla \times A)$~~

consider $\int (A \cdot \delta B) \, dV = \int A \cdot \delta (\nabla \times A) \, dV$

$$= \int A \cdot \nabla \times \delta A \, dV$$

$$= \int \left[\underbrace{\nabla \cdot (\delta A \times A)}_{\text{zero.}} + \delta A \cdot (\nabla \times A) \right] dV = \int \delta A \cdot B \, dV$$

$$\Rightarrow \int \left(B\cdot\delta B + 2\Lambda\, B\cdot\delta A \right) dV = 0$$

$$\Rightarrow \quad \delta B = \Lambda\, \delta A$$

The variation in B should be proportional to variation in A which would imply: $B = \Lambda A$

or $\quad \nabla\times B = \Lambda \nabla\times A = \Lambda B$ — force-free eqn.

So, under Taylor's hypothesis the ~~magno~~ relaxed state of the magnetic field is obtained by the force-fee eqn.

<u>Mathematical aside:</u>

The above proof is mathematically speaking not quite satisfactory. Also it does not make immediate connection to the Euler-Lagrange eqns. which are typically used in ~~min~~ functional minimisation problems. This connection is better ~~mage~~ made in the appendix on functional ~~methods~~ derivatives.

Using the ideas in the appendix we prove the same mathematical theorem again:

$$S[A,B] = \int \mathcal{L}[A,B]\, dV$$

with $\quad \mathcal{L}[A,B] = \dfrac{B^2}{2} + \Lambda A\cdot B = \frac{1}{2} B_k B_k + \Lambda A_k B_k$

$$\frac{\delta S}{\delta A_j(y)} = 0 \quad \text{— minimisation principle.}$$

$$\frac{\delta S}{\delta A_j(y)} = \int \frac{\partial \mathcal{L}}{\partial A_j} \frac{\delta A_j(x)}{\delta A_j(y)}\, d^d x + \cancel{\frac{\partial \mathcal{L}}{\partial B_j}} \int \frac{\partial \mathcal{L}}{\partial B_i} \frac{\delta B_i(x)}{\delta A_j(y)}\, d^d x$$

$$\frac{\delta A_j(x)}{\delta A_j(y)} = \delta(x-y)$$

$$\frac{\delta B_i(x)}{\delta A_j(y)} = \frac{\delta}{\delta A_j} \epsilon_{imn} \partial_m A_n(x)$$

$$= \epsilon_{imn} \partial_m \delta_{jn} \delta(x-y)$$

$$\Rightarrow \quad \frac{\delta S}{\delta A_j} = \frac{\partial \mathcal{L}}{\partial A_j} + \int \frac{\partial \mathcal{L}}{\partial B_i} \epsilon_{imn} \delta_{jn} \partial_m \delta(x-y)\, d^d x$$

$$= \frac{\partial \mathcal{L}}{\partial A_j} - \int \epsilon_{imn} \delta_{jn} \delta(x-y) \partial_m \left(\frac{\partial \mathcal{L}}{\partial B_i}\right) d^d x$$

$$= \frac{\partial \mathcal{L}}{\partial A_j} - \epsilon_{imj} \partial_m \left(\frac{\partial \mathcal{L}}{\partial B_i}\right) = 0$$

$$\frac{\partial \mathcal{L}}{\partial A_j} = \Lambda B_j \qquad \frac{\partial \mathcal{L}}{\partial B_i} = \frac{1}{2} B_i + \Lambda A_i$$

$$\Rightarrow \quad \Lambda B_j - \epsilon_{imj} \partial_m \left(\frac{1}{2} B_i + \Lambda A_i\right) = 0$$

$$\epsilon_{imj} \partial_m A_i = - \epsilon_{jmi} \partial_m A_i = -(\nabla \times A)_j = - B_j$$

$$\epsilon_{imj} \partial_m B_i = - \epsilon_{jmi} \partial_m B_i = -(\nabla \times B)_j$$

$$\Rightarrow \quad 2 \Lambda B = \nabla \times B, \qquad \Rightarrow \quad \boxed{\nabla \times B = \Lambda B}$$

The force-free eqn.

J. Taylor PRL 33 1139 (1974)

- Although the mathematical problem is easily treatable the physical applicability of this theorem is not clear. It may be the case that the energy may decay over a time scale much faster than magnetic helicity. Then, ~~the~~ this ~~formulation~~ can apply to the case ~~where~~ in intermediate time scales.

- It may also be that the problem requires completely different formulation. Typically, minimum energy principle ~~may~~ do not hold for dissipative systems. ~~The case~~ A better principle could be maximisation of some kind of entropy. A surrogate for entropy could be the rate of energy dissipation

$$\epsilon_M = \eta \int J^2 \, dV$$

So, the new extreemisation principle could be:

$$\delta \left(\epsilon_M + \Lambda \mathcal{H} \right) = 0$$

This can give the following expression:

B. Dasgupta et al.

$$\nabla \times \nabla \times \nabla \times B = \Lambda B$$ — PRL 81 3144 (1988)

A subset of which are the force-free equations.

- what is the experimental / numerical evidence?

In certain experimental situations Taylor's hypothesis gives pretty good fit but not in all cases. There are now known to be many exceptions. See section 15.4 of PFP.

- There is another way often use to circumvent the disagreement of the numerical data.

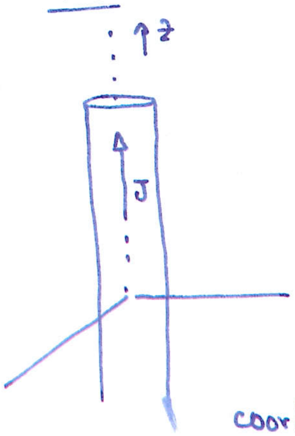
  To propose non-linear ~~force~~- free fields:

$$\nabla \times B = \Lambda B$$

where Λ is no longer a constant but a function of coordinate, often only one.

- See also the illuminating comments on decay of a magnetic field in section 15.1 of PFP.

4.3 Pressure ballanced plasma column:



$$\partial_t (\rho v) + \operatorname{div}(\rho v_i v_j + p\delta_{ij}) = J \times B$$

A steady state can be obtained iff

$$v = 0, \qquad \nabla p = J \times B$$

Consider a column of plasma, use cylindrical coordinates. Assume all quantities are function of r only. We set up a current $J_z(r)$.

$$\mu_0 J = \nabla \times B = -$$

$$= \frac{1}{r}\left(\frac{\partial}{\partial r} r B_\theta - \frac{\partial B_r}{\partial \theta}\right)\hat{z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r} r B_\theta \,\hat{z}$$

$$B = (0, B_\theta(r), 0)$$

$$J \times B = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ 0 & 0 & J_z \\ 0 & B_\theta & 0 \end{vmatrix} = -\hat{r}\, J_z B_\theta = -\frac{\hat{r}}{\mu_0}\frac{1}{r}\frac{d}{dr}(r B_\theta) B_\theta$$

$$\Rightarrow \quad \frac{dp}{dr} = -\frac{1}{r} B_\theta \frac{d}{dr}(r B_\theta)\frac{1}{\mu_0}$$

$$= -\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2 B_\theta^2}{2}\right)\frac{1}{\mu_0}$$

The solution of this can give a ~~pres~~ stationary solution.

If we assume J = constant $= J_0 \hat{z}$

$$\Rightarrow \quad \frac{1}{r}\frac{d}{dr}\left(r B_\theta\right) = J_0 \mu_0 \Rightarrow \quad B_\theta(r) = \frac{r J_0 \mu_0}{2}$$

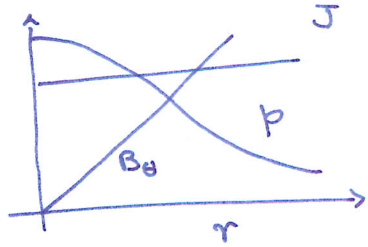
$$\Rightarrow \quad \frac{dp}{dr} = -\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{2}\cdot\frac{r^2}{4} J_0^2 \mu_0^2\right)\frac{1}{\mu_0}$$

$$\cancel{p(r) - p(c) = -\frac{1}{8}\frac{J_0^2 r^5}{r^2\, 5}} = -\frac{J_0^2 r^3}{\cancel{40}}$$

$$= -\frac{1}{r^2}\frac{1}{\mu_0}\frac{d}{dr}\left(\frac{r^4}{8} J_0^2 \mu_0^2\right) J_0^2$$

$$= -\frac{\mu_0}{\cancel{8}\,2}\frac{1}{r^2}\cancel{4}\cdot r^{\cancel{3}} J_0^2$$

$$\Rightarrow \quad p(r) - p(0) = -\frac{\mu_0 J_0^2 r^2}{4}$$



pressure decrease outward!

At a distance $r = a$, $p = 0$

$$\Rightarrow \quad p(0) = \frac{\mu_0 J_0^2}{4} a^2$$

Given a J_0, and a pressure at $r=0$, The pressure can become 0 at a radius $r = a$. For $r > a$ the pressure can become negative? This is ~~clarly~~ clearly unphysical. But this shows what a "pinch" is.

4.4 The solar wind.

Consider a central star. We look for a spherically symmetric solution of the equations of hydrodynamics (not MHD)

$$\vec{v} = (v_r(r), 0, 0) \quad ; \qquad \rho = \rho(r)$$

$$c_s^2 = \frac{p}{\rho} = \text{constant}$$

Also, assume that the solutions is stationary.

$$\partial_t \rho = 0, \qquad \partial_t (\rho v) = 0$$

In spherical polar coordinate system: 

$$\cancelto{0}{\partial_t \rho} + \mathrm{div}\,(\rho v_r) = 0$$

$$\frac{1}{r^2} \frac{d}{dr}\left(r^2 \rho v_r\right) = 0 \quad \Rightarrow \quad \underbrace{r^2 \rho v_r}_{\text{rate of mass injection}} = \dot{M} = \text{constant}$$

$$\cancelto{0}{\partial_t(\rho v_r)} + \rho v_r \frac{dv_r}{dr} = -\frac{dp}{dr} - \frac{GM}{r^2}\rho \quad \leftarrow \text{gravity force.}$$

$$\Rightarrow \quad \frac{d}{dr}\left(\frac{v_r^2}{2} + c^2 \ln \rho - \frac{GM}{r}\right) = 0$$

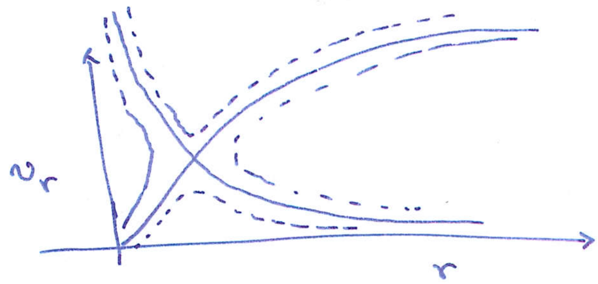
$$\Rightarrow \quad \frac{v_r^2}{2} + c^2 \ln \rho - \frac{GM}{r} = \text{constant} \equiv \mathcal{E} \quad \leftarrow \text{Energy conservation}$$

combining:

$$\boxed{-\frac{GM}{r} + \frac{v_r^2}{2} - c^2 \ln v_r - 2c^2 \ln r = \mathcal{E}'}$$

The Parker solution.

- To find the actual ~~numbe~~ solution we have to numerically solve the transcendental eqn. This shows that for a given ~~bou~~ parameters there are four branches



- The solution can be negative, i.e. accretion, or positive : i.e. wind. Numerically each branch is found separately and then connected by hand.

- It is remarkable how little assumptions we need to obtain the wind. The pressure goes to zero at infinity.

- A different way to write the same eqn.

$$\left(v_r^2 - c^2\right) \frac{d}{dr} \left| \ln v_r \right| = \frac{2c^2}{r} - \frac{GM}{r^2}$$

Assume that at $r = r_*$, $v_r = c$

$$\Rightarrow \quad \frac{2c^2}{r_*} = \frac{GM}{r_*^2} \quad \Rightarrow \quad r_* = \frac{GM}{2c^2}$$

The radius at which the wind becomes supersonic.

- The problem can also be easily extended to adiabatic wind.

- This is a steady wind, in the sense that $v_r(r)$ is not a function of time but the ~~solar~~ ~~wind~~ actual solar wind is a highly turbulent process with huge fluctuations.

- The mass loss due to the wind is quite small.

- Also consider the magnetized wind. As we move ~~aga~~ away from the star the velocity of the wind increases, and in the present solution the energy is conserved. If we include magnetic field then total energy (magnetic + kinetic + gravitational + ..) will be conserved. Far away from the star the magnetic energy must decrease. Hence the kinetic energy must increase. The point upto which the (magnetic energy) > kinetic energy is called Alfven radius.

Appendix 4A

Functional Calculus

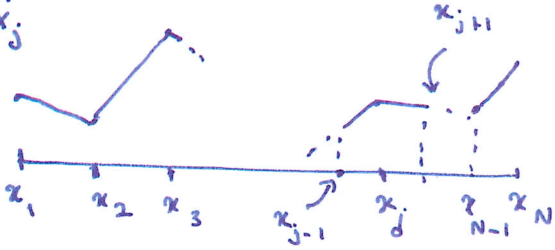
~~consider a function of many variable~~ :

$$f(\{x_i\})$$

~~How to find the minima/maxima of the function~~ ?

$$\cancel{\nabla f = \frac{\partial f}{\partial x_j}\hat{x}_j}$$

consider a disreete space



At each x_i, consider a height function $h(x_j)$.

Then consider a function of $h(x_j)$; for example:

$$\tilde{\Phi}[h_j] \equiv \tilde{\Phi}[h(x_j)]$$

This is a function of N variables; h_j not a functional.

specifically this has a Taylor series expansion

$$\tilde{\Phi}(h_j + \delta h_j) = \tilde{\Phi}(h_j) + \sum \frac{\partial \tilde{\Phi}}{\partial h_j}\delta h_j + \frac{1}{2}\sum \frac{\partial^2 \tilde{\Phi}}{\partial h_j \partial h_k}\delta h_j \delta h_k$$

and

$$\frac{\partial \tilde{\Phi}}{\partial h_j} = \lim_{\delta h_j \to 0} \frac{\tilde{\Phi}(h_1, \ldots, h_j + \delta h_j, \ldots h_N) - \tilde{\Phi}(h_1, \ldots h_N)}{\delta h}$$

Now to calculate the functional take the continuum limit $\lim_{N\to\infty}$, $\lim_{\delta x \to 0}$ and replace the sum

$$\sum \to \sum \Delta x \to \int dx$$

$$\Phi[h+\delta h] = \Phi[h] + \int dx \frac{\delta \Phi}{\delta x} + \frac{1}{2}\int dx\, dx' \frac{\delta^2 \Phi}{\delta h(x)\delta h(x')}$$

$$+ \cdots$$

In particular

$$\frac{\delta h(x)}{\delta h(y)} = \delta(x-y)$$

$$\frac{\delta f(h(x))}{\delta h(y)} = \frac{df(z)}{dz} \cdot \frac{\delta h(x)}{\delta h(y)} = f' \delta(x-y)$$

— usual chain rule of derivatives.

Now consider

$$\Phi[h] = \int f\left(h, \frac{dh}{dx}\right) dx = \int f(h, h')\, dx$$

is a function of h and it's derivative.

$$\Rightarrow \frac{\delta \Phi}{\delta h(y)} = \int \frac{\partial f}{\partial h} \frac{\delta h(x)}{\delta h(y)} dx + \int \frac{\partial f}{\partial h'} \frac{\delta h'(x)}{\delta h(y)} dx$$

$$= \int \frac{\partial f}{\partial h} \delta(x-y)\, dx + \int \frac{\partial f}{\partial h'} \cdot \frac{\delta}{\delta h(y)} \left(\frac{dh}{dx}\right) dx$$

$$= \frac{\partial f}{\partial h} + \int \frac{\partial f}{\partial h'} \frac{d}{dx} \delta(x-y)\, dx$$

$$= \frac{\partial f}{\partial h} - \int \delta(x-y) \frac{d}{dx} \frac{\partial f}{\partial h'} dx + \left(\begin{array}{l}\text{terms zero} \\ \text{in integration} \\ \text{by parts}\end{array}\right)$$

$$= \frac{\partial f}{\partial h(y)} - \frac{d}{dy} \frac{\partial f}{\partial h'}$$

$\frac{\delta \Phi}{\delta h} = 0$ typically gives us the Euler-Lagrange eqn.

$$\boxed{\frac{\partial f}{\partial h} - \frac{d}{dy} \frac{\partial f}{\partial h'} = 0}$$

In higher dimensions, we get $\boxed{\frac{\partial f}{\partial h} - \nabla \cdot \frac{\partial f}{\partial \nabla h} = 0}$

Examples

1. Simplest from Lagrangian mechanics.

A particle in a one dimensional potential.

The Newton's eqn gives the eqn. of motion to be:

$$\cancel{m\frac{dv}{dt}} = \frac{\partial V}{\cancel{\partial x}} \cancel{\nabla V}$$

$$\boxed{\cancel{m\frac{dv}{dt} = -\frac{\partial V}{\partial x}}} \qquad \frac{d}{dt}(m\dot{x}) = -\frac{\partial V}{\partial x}$$

The Lagrangian mechanics ~~th~~ states that the action

$$S = \int_{t_1}^{t_2} \mathcal{L}(x,\dot{x})\,dt$$

will reach ~~ah~~ a minima.

$$\Rightarrow \cancel{\frac{\delta S}{\delta x}} \cancel{= 0} = \int_{t_1}^{t_2} \mathcal{L}(x,\dot{x})\,dt \;, \quad \frac{\delta S}{\delta x} = 0$$

The corresponding Euler-Lagrange eqn. is:

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - V(x)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

$$\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial V}{\partial x}$$

$$\frac{\partial \mathcal{L}}{\partial t} = 0$$

$$\boxed{\frac{d}{dt}(m\dot{x}) = -\frac{\partial V}{\partial x}}$$

True for all ~~x(t)~~ $x(t)$

Exercise II

To be returned on $\boxed{\text{9th Feb 2016 Tuesday}}$

1. show that $\chi(r) = \frac{1}{4} r^2 + Ar + \frac{B}{r}$

is a solution of the biharmonic eqn. $\nabla^4 \chi = 0$

where r is the radial coordinate in spherical polar coordinate system.

A and B must be chosen such that

$$u_i = U_{ij} U_j \quad \text{with} \quad U_{ij} = \delta_{ij} \nabla^2 \chi - \frac{\partial^2 \chi}{\partial x_i \partial x_j}$$

must be zero at $r = 1$.

Show that this boundary condition implies $\chi''(1) = \chi'(1) = 0$

From this, show that $B = \frac{1}{4}$, $A = 3/4$

Then show that for $r > 1$

$$\vec{u} = \vec{U} - \frac{3}{4}\left(\frac{\vec{U}\cdot\vec{r}\,\vec{r}}{r^3} + \frac{\vec{U}}{r}\right) - \frac{1}{4}\nabla\left(\frac{\vec{U}\cdot\vec{r}}{r^3}\right)$$

and $$p = -\frac{3}{2}\frac{\vec{U}\cdot\vec{r}}{r^3}$$

5

2. Write down the equations of isothermal hydrodynamics. Then non-dimensionalize the equations. Assume that there is a characteristic length scale $l \equiv \frac{1}{k_f}$, velocity scale u, constant sound speed c, ~~and~~ a magnetic field B_0, and a constant background density ρ_0. ~~So~~ show that the non-dimensionalized equations have the following parameters

$$Re = \frac{u}{\nu k_f}, \quad Re_M = \frac{u}{\eta k_f}, \quad Ma = \frac{u}{c},$$

$$Ma_A = \frac{u}{c_A}, \quad c_A = \frac{B_0}{(\rho_0 \mu_0)^{1/2}}$$

~~2/10~~.

3. Functional derivatives : (5)

Consider a relativistic particle in free spac. ~~The Lagrangian is given by~~
The action is given by

$$S = -\int mc^2 \left(1 - \frac{\dot{x}^2}{c^2}\right)^{1/2} dt$$ (5) ~~2/10~~.

By taking functional derivatives (use the theorem proved in class) to find out the equation of motion of a free relativistic particle.

4. Work out the solution of the solar wind problem assuming the flow to be adiabatic (instead of isothermal as was worked in class)

(5) ~~2/10~~.

<u>Total 20 credits</u>

4.5

~~The Sedov - Taylor problem~~:

Pressure ballanced plasma column

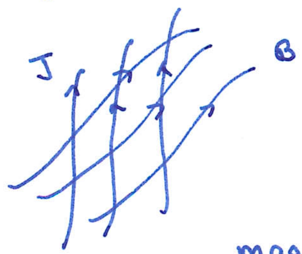
$$\mathrm{div}\, B = 0$$

$$\mathrm{div}\left(p\,\delta_{ij}\right) = J \times B \qquad v = 0, \quad \rho = \text{constant}$$

$$\nabla \times B = \mu_0 J$$

$$\Rightarrow \quad (B \cdot \nabla)\, p = 0 \quad \text{and} \quad (J \cdot \nabla)\, p = 0$$

pressure gradient is zero (pressure is constant) along lines of constant B and constant J



Hence $p(x, y, z) = \text{constant}$ are surfaces which contain magnetic lines of force and lines of J.

These are called magnetic surfaces. Every magnetic surface could be boundary of an equilibrium configuration.

$$\Rightarrow \quad \mathrm{div}\left(p\,\delta_{ij} + \frac{1}{\mu_0}\frac{B^2}{2}\delta_{ij} + \frac{1}{\mu_0} B_i B_j\right) = 0$$

$$\mathrm{div}\, \Pi_{ij} = 0$$

~~consider~~ $\cancel{x_j \Pi_{ij}\, dv}$

consider $\int x_i \partial_k \Pi_{ik} \, dv$

$$= \int \partial_k \left(\Pi_{ik} x_i \right) - \left(\Pi_{ik} \partial_k x_i \right) dv$$

$$= \int \partial_k \left(\Pi_{ik} x_i \right) dv - \int \Pi_{ii} \, dv$$

$= 0$

$$\Rightarrow \int \Pi_{ii} \, dv = \int \partial_k \left(\Pi_{ik} x_i \right) dv$$

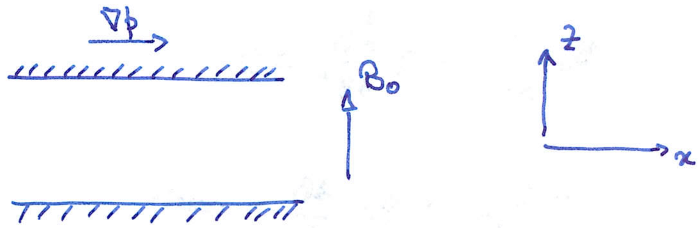
$$= \oint_S \Pi_{ik} x_i \hat{n}_k \, ds$$ - Gauss's theorem.

$$\Rightarrow \int 3 \left(p + \frac{1}{\mu_0} \frac{B^2}{2} \right) dv = \oint_S \Pi_{ik} x_i \hat{n} \, ds$$

Let the plasma be confined by the surface $p = 0$ which does not extend to infinity. Then taking S at infinity the RHS $= 0$. But the LHS is never zero.

$\Rightarrow$ ~~Equi~~ steady state configuration of plasma in finite space is possible only if there are ~~currents at infinity~~. ~~so~~ external sources of currents.

Hartmann flow :



Pressure driven flow of plasma.

symmetry : v_x is non-zero. and function of z only.

B_x is non-zero and function of z only

p is function of x only.

steady flow $\partial_t v = 0$

$\partial_t \rho = 0$

$\partial_t B = 0$

also $\operatorname{div} B = 0$

Incompressible : $\operatorname{div} v = 0$

$$(v\cdot\nabla)\, v = \nu \nabla^2 v - \nabla p + J\times B$$

$$J = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ B_x & 0 & 0 \end{vmatrix} = \partial_z B_x \hat{y}$$

LHS : $\cancel{v_x\partial}\ \ \partial (v_x \partial_x)\, v_x = 0$

$\Rightarrow$ $\hat{x}\, \nu \dfrac{d^2}{dz^2} v_x = \partial_x p \cancel{+} - J\times B$

$$= +\partial_x p\,\hat{x} + \hat{z}\, B_x \frac{d}{dz}\frac{B_x}{\mu_0} +$$

$$\partial_z p\,\hat{z} \;\bar{+}\; \hat{x}\, B_0 \frac{d}{dz}\frac{B_x}{\mu_0}$$

$$J\times B = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \partial_z B_x & 0 \\ B_x & 0 & B_0 \end{vmatrix} = -\hat{z}\, B_x \partial_z B_x \quad \hat{x}\, B_0 \partial_z B_x$$

$\Rightarrow$ $p + \dfrac{1}{2}\dfrac{B_x^2}{\mu_0} =$ function of x only.

Also : $\nu \dfrac{d^2 v_x}{dz^2} + \dfrac{B_0}{\mu_0}\dfrac{dB_x}{dz} = \partial_x p = \text{const}$

Induction eqn :

$$\nabla \times (v \times B - \eta J) = 0$$

$$\Rightarrow \quad \nabla \times (v \times B) + \frac{\eta}{\mu_0} \nabla^2 B = 0$$

$$\Rightarrow \quad B_0 \frac{d v_x}{d z} + \eta \frac{d^2 B}{d z^2} = 0$$

Boundary condition :

$v_x = 0$ at the boundary

$B_x = 0$

$J_y = 0$

At the end :

$$\frac{d^2 v_x}{d z^2} - \frac{1}{\delta^2} v_x + \Lambda = 0$$

with $\delta = \frac{\sqrt{\nu \eta \mu_0}}{B_0}$

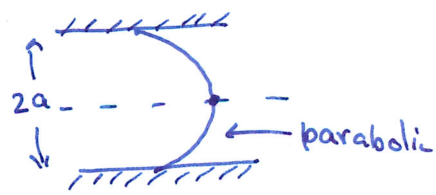
solution :

$$v = v_0 \frac{\cosh(a/\delta) - \cosh(z/\delta)}{\cosh(a/\delta) - 1}$$

The Hartmann number $G = \frac{a}{\delta}$. ← new length scale.

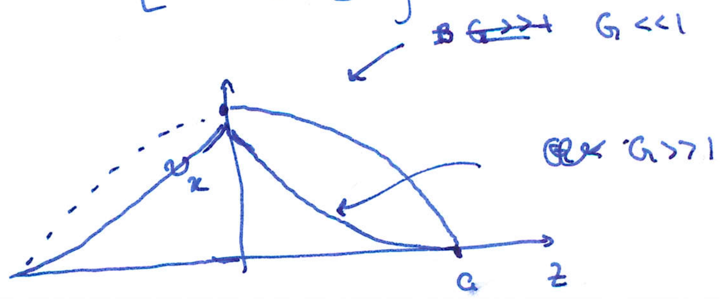
For $G \ll 1$, $\Rightarrow$ small magnetic field :

$$v = v_0 \left(1 - z^2/a^2\right)$$



For $G \gg 1$

$$v = v_0 \left\{ 1 - \exp\left[- \frac{a - |z|}{\delta} \right] \right\}$$



Lecture V

Linear stability

5.1 Consider a dynamical system with N degrees of freedom, $x_1, \ldots x_j, \ldots x_N$, & given by a state vector $|x\rangle$. Let the dynamical system be described by the following ~~partial~~ partial differential equation:

$$\partial_t |x\rangle = \mathcal{N}[|x\rangle]$$

where $\mathcal{N}$ is in general any non-linear function of $|x\rangle$, and its spatial derivatives.

Then assume that this system of equation have a solution $|x_0\rangle$ such that;

$$\partial_t |x_0\rangle = \mathcal{N}[|x_0\rangle]$$

But in general $|x_0\rangle$ may be just one of ~~in~~ infinite number of possible solutions.

Is $|x_0\rangle$ stable ?

Let us qualify the question. If we make a small change to $|x_0\rangle$; $|\delta x\rangle$, then we can write

$$\partial_t\left[|x_0\rangle + |\delta x\rangle\right] = \mathcal{N}\left[|x_0\rangle + |\delta x\rangle\right]$$

If $|\delta x\rangle$ is small, and the function $\mathcal{N}$ can be Taylor expanded at $|x_0\rangle$, we can write

$$\cancel{\partial_t |x_0\rangle} + \partial_t |\delta x\rangle = \cancel{\mathcal{N}[|x_0\rangle]} + \left.\frac{\delta \mathcal{N}}{\delta x}\right|_{|x_0\rangle} |\delta x\rangle + \text{h.o.t.}$$

~~Let~~ The operator $\left.\frac{\delta \mathcal{N}}{\delta x}\right|_{|x_0\rangle} |\delta x\rangle$ is linear in $|\delta x\rangle$. So, upto leading order in $|\delta x\rangle$ we can write

$$\boxed{\partial_t |\delta x\rangle = \mathcal{L}[x_0]\,|\delta x\rangle}$$

where $\mathcal{L}[x_0]$ is the linearized operator of $\mathcal{N}$ about $|x_0\rangle$.

This equation is a linear equation; subject to the same boundary conditions as the original equations. Hence can often be solved in N-dimensional vector space. This problem is far easier to solve than the original non-linear problem.

- It may not always be tractable analytically but can often be solved numerically.

- By solving this problem we can find out whether ~~to~~ the perturbation $|\delta x\rangle$ shall grow in time or not. Often time-dependence of $|\delta x\rangle$ is written as $\exp(i\omega t)$.

Let $\omega = \omega_R + i\,\omega_I$

$\Rightarrow \exp(i\omega t) \sim e^{i\omega_R t}\, e^{-\omega_I t}$

If $\omega_I > 0$, $|\delta x\rangle$ decays in time
$\omega_I < 0$ $|\delta x\rangle$ grows in time
$\omega_I = 0$ $|\delta x\rangle$ has wave-like behavior.

This solves the stability problem, BUT only upto leading order in perturbation theory; hence only applies to infinitismal perturbations which may not be physically relevant. We shall come back to this point later.

5.2 A simple case :

Linear stability step I : find a solution to the dynamical eqns.

Finding a solution itself may be non-trivial. Also, if the solution itself is time-dependent then its stability analysis can be quite complicated.

So let us start with a simple time-independent solution.

Equations of isothermal MHD :

$$\partial_t(\rho v) + \mathrm{div}(\rho v_i v_j + p\delta_{ij} + \sigma_{ij}) = J \times B$$

$$J\times B \to -\frac{1}{2}\nabla B^2 \frac{1}{\mu_0}, \quad (B\cdot\nabla)B\frac{1}{\mu_0}$$

$$\partial_t \rho + \mathrm{div}(\rho v) = 0 \qquad c^2 =$$

$$\partial_t B = \nabla \times (v \times B - \eta J)$$

<u>solution</u> $\quad v = 0, \quad \rho = \text{constant} = \rho_0, \quad B = \hat{z}B_0 = \text{constant}$

<u>Linear stability step II</u> : Add an infinitismal perturbation and linearize the equations :

$$v = v_0 + \delta v, \qquad \rho_0 \partial_t \delta v = -c^2 \nabla \delta\rho - \nabla\left(\frac{B_0\cdot\delta b}{\mu_0}\right) + \frac{1}{\mu_0}(B_0\cdot\nabla)\delta b$$

$$\rho = \rho_0 + \delta\rho,$$

$$B = B_0 + \delta b, \qquad \partial_t \delta\rho = -\rho_0(\nabla\cdot\delta v)$$

$$\partial_t \delta b = (B_0\cdot\nabla)\delta v - B_0(\nabla\cdot\delta v)$$

(ignoring dissipative terms which are linear in δv, and δb)

This is a matrix equation if we use Fourier transforms:

$$~~\delta u~~ \quad \delta v = \tilde{v} \exp(i\omega t + i k \cdot x)$$

$$\delta \rho = \tilde{\rho} \exp(i\omega t + i k \cdot x)$$

$$\delta b = \tilde{b} \exp(i\omega t + i k \cdot x)$$

substituting we get the form:

$$M|\psi\rangle = 0$$

where $|\psi\rangle = (\tilde{v}_x, \tilde{v}_y, \tilde{v}_z, \tilde{\rho}, \tilde{b}_x, \tilde{b}_y, \tilde{b}_z)$

$$M = \begin{pmatrix} i\omega & \times & \times & \times & & & \\ & i\omega & \times & \times & & & \\ & & \cdot & & & & \\ \times & \times & & \cdot & & & \\ & & & & \cdot & & \\ \times & \times & & & & \cdot & \\ & & & & & & i\omega \\ \times & \times & & & & & \end{pmatrix}$$

A solution exists only when $\det M = 0$

That would be $f(\omega, k) = 0$

which is ~~an~~ ~~ad~~ a function polynomial in ω, with a 7th order polynomial; in other words there are 7 branches of the (ω, k) relationship. These are called dispersion relations.

Let us make life particularly simple:

space in 1-dimensional, along z; and $B_0 = B_0 \hat{z}$

$\Rightarrow \quad \partial_x = 0, \quad \partial_y = 0$

Then we obtain the following solution for ω.

$\omega = 0$, (one branch)

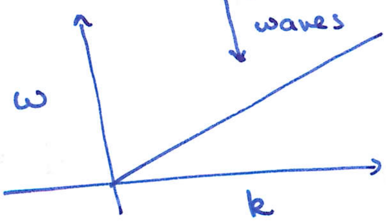
$\omega^2 - c^2 k^2 = 0$ (two degenerate branches)

$\omega^4 - 2 v_A^2 k^2 \omega^2 + v_A^4 k^4 = 0$

$(\omega^2 - v_A^2 k^2)^2 = 0$

four degenerate branches.

$$v_A = \frac{B_0}{(\rho_0 \mu_0)^{1/2}}$$



$\boxed{\text{Alfven waves}}$

- Furthermore the Alfvenic modes are perpendicular to the direction of propagation $\hat{z}$. They are transverse waves. Magnetic field behaves like a string.

- If we look at incompressible approximation; then $\Longleftrightarrow$ div $\delta v = 0$

$\Rightarrow \quad i k_\mu \tilde{v}_\mu = 0 \quad \Rightarrow \quad k_z v_z = 0$

$\Rightarrow \quad \tilde{v}_z = 0$ and $\tilde{\rho} = 0$, But the Alfven wave survives. Although sound waves do not.

- How large can the amplitude of the waves be? There is no limit from linear theory, they can be anything.

5.3 The more general case of 3-d space:
(But keep the equations isentropic)

The linearized equations look like:

$$\omega \vec{b} = \vec{k} \times (\vec{v} \times \vec{B})$$

$\Rightarrow$ The perturbations in $\vec{b}$ are perpendicular to the direction of propagation of the wave.
They are transverse waves.

Take $\vec{k}$ along the x direction.
Also ~~noth~~ note that $\vec{k} \cdot \vec{b} = 0$ is automatically satisfied.
and define phase velocity $u = \frac{\omega}{k}$
The the x-y plane as the plane containing $\vec{k}$ and $\vec{B}$.

Then we have:

$$u b_z = - v_z B_x \qquad u v_z = - \frac{B_x b_z}{\mu_0 \rho_0}$$

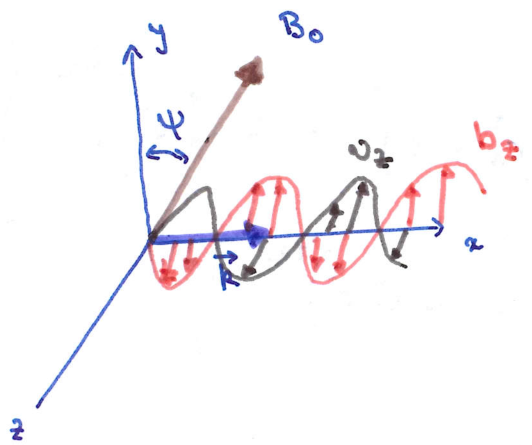
This two forms a pair.
For both of them to be true: $u^2 = \frac{B_x^2}{\mu_0 \rho_0}$

$\Rightarrow$ $\boxed{\omega = c_A k}$. $c_A = \frac{B_x}{\sqrt{\mu_0 \rho_0}}$

We get the Alfven result back again.

But with B_x

and $b_z = - \sqrt{\mu_0 \rho_0}\, v_z$



Now consider the other two components

$$u b_y = v_x B_y - v_y B_x$$

$$u v_y = -\frac{B_x b_y}{\mu_0 \rho_0}$$

$$v_x \left(u - \frac{c^2}{u} \right) = B_y b_y / \mu_0 \rho_0$$

By setting the determinant to zero we have:

$$u_{f,s}^2 = \frac{1}{2} \left\{ \frac{B^2}{\rho_0 \mu_0} + c^2 \pm \left[\left(\frac{B^2}{\mu_0 \rho_0} + c^2 \right)^2 - 4 \frac{B_x^2}{\mu_0 \rho_0} c^2 \right]^{1/2} \right\}$$

fast and slow magnetosonic waves.

<u>Case 1</u> $\quad \frac{B^2}{\mu_0 \rho_0} << c^2$ $\quad$ small magnetic energy compared to the energy in sound modes.

$u_f \simeq c \Rightarrow v_y << v_x$

we essentially get back usual sound waves.

$u_s \simeq c_A \Rightarrow v_x << 1,$

$\Rightarrow c_A b_y \simeq -v_y B_x$

$\Rightarrow \cancel{v_y} \simeq -\frac{b_y}{}$

$\Rightarrow b_y \simeq -\sqrt{\mu_0 \rho_0}\, v_y$

Alfven like. But different polarization.

<u>case 2</u> $\frac{B^2}{\mu_0 \rho_0} >> c^2$

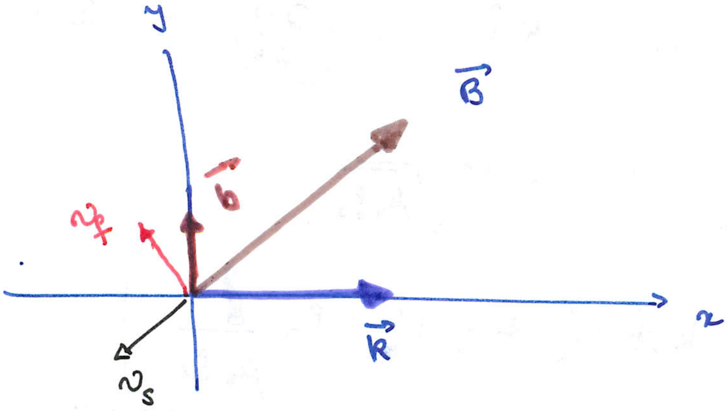
$\Rightarrow \quad u_f \sim \frac{B}{\sqrt{\mu_0 \rho_0}}$

and $\left(v_x B_x + v_y B_y\right) \sim 0$

$\Rightarrow$ $\vec{v}$ is perpendicular to $\vec{B}$.

$u_s \sim c \frac{B_x}{B}$

and $\vec{v}$ is anti-parallel to $\vec{B}$



$u_s \leq c_A \leq u_f$, $u_f \geq c$, $u_s \leq c$

If $\vec{k}$ and $\vec{B}$ are parallel: we have, c_A and c

for $\vec{k}$ and $\vec{B}$ are perpendicular: only fast magnetosonic waves exists.

5.4 Effects of dissipative terms :

clearly the solution will loose energy and the waves will damp as they progress.

Let $\langle Q \rangle \equiv$ average energy dissipation rate

$$\simeq \eta J^2 + \nu \omega^2 \rho_0$$

$$\simeq \eta \left(\frac{\partial b}{\partial x}\right)^2 + \nu \left(\frac{\partial v}{\partial x}\right)^2 \rho_0$$

$\langle q \rangle$ = average energy flux

$$\simeq - B_x \, b \cdot v$$

Both of them are quadratic in ~~the~~ fluctuations which is the first non-zero correction.

As the wave progresses

$$(\text{wave energy}) \sim e^{-\langle Q \rangle / \langle q \rangle \, x}$$

$$(\text{wave amplitude}) \sim e^{-\langle Q \rangle / 2\langle q \rangle \, x}$$

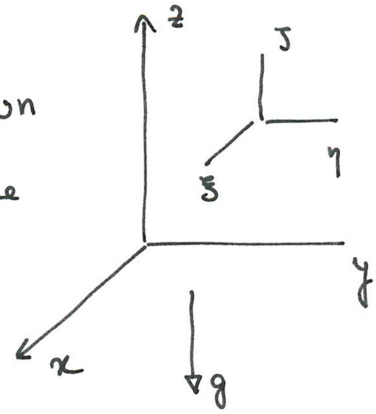
From Alfven waves $\sim e^{-\gamma x}$

with $$\gamma = \frac{\omega^2}{2 C_A^2} \left(\eta + \frac{\nu}{\mu_0} \right)$$

Exercise III   Return on 16th Feb 2016

1. Wave equation from ~~as~~ a Lagrangian standpoint. :

Consider a vertically stratified compressible fluid at rest in gravity.



The coordinate system is as shown in the figure. (ξ, η, ζ) are the Lagrangian displacements.

The kinetic energy density

$$T = \frac{1}{2} \rho \left(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 \right)$$

The potential energy is the sum of several terms :

(a) Elastic energy : $V_{el} = \frac{1}{2} \lambda \epsilon^2$

with $\epsilon = \left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} \right)$, $\lambda = \rho c^2$ is the bulk modulus

(b) If you move a fluid parcel by ζ upward, there is a difference in density between the ~~parcel~~ parcel and its new surrounding.

$$\Delta \rho = \rho_0 (z + \zeta) - \rho_0 (\zeta)$$

$$= \left(\frac{d\rho_0}{dz} \right) \zeta$$

The buoyancy force

$$F = -g\,\Delta\rho = -g\left(\frac{d\rho_0}{dz}\right)\zeta$$

The corresponding potential:

$$V_B = -\frac{1}{2}\,g\left(\frac{d\rho_0}{dz}\right)\zeta^2$$

(c) There is a third contribution:

As the particle is displaced upward; it is also compressed. This compression is due to the compressive field

$$\epsilon = \left(\partial_x\xi + \partial_y\eta + \partial_z\zeta\right)$$

The corresponding potential energy is

$$V_c = \zeta\,g\,\Delta\rho$$

$$= \zeta g(-\rho\epsilon)$$

$$= -\rho g\zeta\epsilon$$

The net Lagrangian

$$\mathcal{L} = T - V_{el} - V_B - V_c.$$

• Given this Lagrangian, show by taking functional derivatives that the corresponding Euler-Lagrange eqn. are:

$$\rho \ddot{\xi} - \frac{\partial}{\partial x} \lambda \epsilon + \rho g \frac{\partial \zeta}{\partial x} = 0$$

$$\rho \ddot{\eta} - \frac{\partial}{\partial y} \lambda \epsilon + \rho g \frac{\partial \zeta}{\partial y} = 0$$

$$\rho \ddot{\zeta} - \frac{\partial}{\partial z} \lambda \epsilon - g\rho\left(\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y}\right) = 0$$

5 marks

2. Write down the linearized equations in a vertically stratified medium.
Then use the Lagrangian displacement ~~to~~ to rewrite the equations.

show that

$$\tilde{\rho} + \rho_0 \partial_x \xi_1 + \rho_0 \partial_y \xi_2 + \frac{d}{dz}(\rho_0 \xi_3) = 0 \qquad -(1)$$

Here $\tilde{\rho}$ is the perturbed density and (ξ_1, ξ_2, ξ_3) are the three Lagrangian displacements.

Similarly, from the linearized momentum eqn.

show that

$$\cancel{\rho_0 \partial_t^2 \xi_3 - \frac{d}{dz} \rho \xi_3}$$

$$\rho_0 \partial_t^2 \xi_3 + \partial_z\left(c^2 \tilde{\rho}\right) = -g\tilde{\rho} \qquad -(2)$$

In (1) ignore $\partial_x \xi_1$ and $\partial_y \xi_2$.

Substitute from (1) to (2) then do the following approximation

$$\frac{\partial \xi_3}{\partial z} \simeq 0$$

Under ~~this~~ these approximations show that the following holds:

5 marks

$$\partial_t^2 \xi_3 + N^2 \xi_3 = 0$$

where $N^2 = -g\left(\frac{d}{dz} \ln \rho_0 + \frac{g}{c^2}\right)$

N^2 is called the Brunt-Väissala frequency.

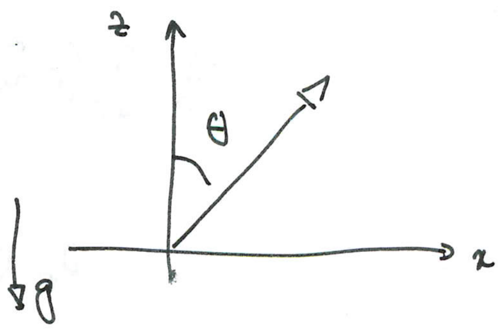
3. In a stratified fluid of uniform Brunt-Vaisalla frequency N^2 show that the equations

$$\tilde{p} = N \sin\theta \, \frac{q_1}{k} \exp\left[i\left(Nt\cos\theta - kx + kz\tan\theta\right)\right]$$

$$\rho_0 u = \left(\tan\theta, 0, 1\right) q_1 \exp\left[i\left(Nt\cos\theta - kx + kz\tan\theta\right)\right]$$

with $0 < \theta < \frac{\pi}{2}$ represents plane internal waves which transmit an energy flux

$$\frac{1}{2}\frac{q_1^2}{\rho_0 k_1} N\tan\theta$$ in a direction shown in the figure



5 marks

4. Stability of inviscid Couette flow:

Consider the inviscid Couette flow with

$$\cancel{U_r} \quad U_r = U_z = 0 \quad \text{and} \quad U_\theta = V(r) = r\,\Omega(r)$$

Here $V(r)$ [and $\Omega(r)$] is an arbitrary function of r.

(a) Write down the linearized equations for the perturbations, assuming axisymmetric perturbation (i.e. δv_r, δv_θ, δv_z, δp are ~~not~~ ~~cons~~ not functions of the angular variable θ)

(b) Assume the perturbations have the following dependence $\sim \exp i(pt + kz)$. Show that the eigenvalue problem is

$$\cancel{i\sigma\,\delta v_r}$$

$$ip\,\delta\hat{v}_r - 2\Omega\,\delta\hat{v}_\theta = -\frac{d}{dr}(\delta\hat{p})$$

$$ip\,\delta\hat{v}_\theta + \left[\Omega + \frac{d}{dr}(r\Omega)\right]\delta\hat{v}_r = 0$$

<u>5 marks</u>

$$ip\,\delta v_z = -ik(\delta\hat{p})$$

and

$$\frac{d\,\delta v_r}{dr} + \frac{\delta v_r}{r} + ik\,\delta v_z = 0$$

(c) Now use Lagrangian dispacements

$$\delta u_r = ip\xi_r\,, \quad \delta v_\theta = ip\xi_\theta - r\frac{d\Omega}{dr}\xi_r\,, \quad \delta v_z = ip\xi_z$$

to show that the following eqn. holds

$$\left[p^2 - \Phi(r)\right]\xi_r = \frac{d\,\delta\hat{p}}{dr}$$

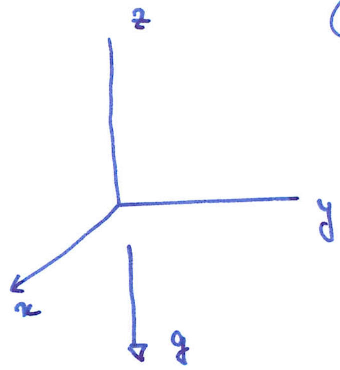
$$\frac{1}{r}\frac{d}{dr}(r\xi_r) = \frac{k^2}{p^2}\,\delta\hat{p}$$

with

$$\Phi(r) = \frac{2\Omega}{r}\frac{d}{dr}\left(r^2\Omega\right)$$

~~lecture~~

Lecture VI



6.1 Internal gravity waves:

waves in a stratified medium

stationary solution:

$$\frac{dp_0}{dz} = -\rho_0 g$$

$$v_0 = 0$$

$$p_0 = p_0(\rho)$$

$$c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s$$

is not a constant any more.

Linearized eqns.

$$\rho = \rho_0 + \tilde{\rho}$$

$$v = v_0 + \tilde{v}$$

$$p = p_0 + \tilde{p},$$

$$\tilde{p} = \left.\frac{\partial p}{\partial \rho}\right|_s \tilde{\rho} = c^2(z)\tilde{\rho}$$

continuity eqn: $\partial_t \rho + \mathrm{div}(\rho v) = 0$

Linearized form: $\boxed{\partial_t \tilde{\rho} + \mathrm{div}(\rho_0 \tilde{v}) = 0}$

momentum eqn: $\partial_t(\rho v) + \mathrm{div}(\rho v_i v_j + p\delta_{ij}) = -\hat{z} g \rho$

Linearized form $\partial_t(\rho_0 \tilde{v}) + \nabla \tilde{p} = -\hat{z} g \tilde{\rho}$

$$\Rightarrow \boxed{\partial_t(\rho_0 \tilde{v}) + \nabla(c^2 \tilde{\rho}) = -\hat{z} g \tilde{\rho}}$$

It is useful to consider the Lagrangian displacement. ξ

such that $\tilde{v} = D_t \xi = (\partial_t + \tilde{v}\cdot\nabla)\xi = \partial_t \xi$

Linearization

$$\Rightarrow \quad \partial_t \tilde{\rho} + \mathrm{div}(\rho_0 \partial_t \xi) = 0$$

Integrating $\partial_t \tilde{\rho} + \mathrm{div}(\rho_0 \xi) = 0$ with the condition $\tilde{\rho} = 0$, for $\xi = 0$

substituting back in the linearized momentum eqn.

$$\rho_0 \partial_t^2 \xi - \nabla\left[c^2 \operatorname{div}(\rho_0 \xi)\right] = + \hat{z}\, g \operatorname{div}(\rho_0 \xi)$$

An eqn ~~quadratic~~ containing second order derivative of both space and time. The problem of understanding the solutions come from the inhomogeneity of the problem.

Expanding in three coordinate directions:

$$\rho_0 \partial_t^2 \xi_1 - \frac{\partial}{\partial x}\left(\rho_0 c^2 \operatorname{div} \xi\right) + \rho_0 g \frac{\partial \xi_3}{\partial x} = 0$$

$$\rho_0 \partial_t^2 \xi_2 - \frac{\partial}{\partial y}\left(\rho_0 c^2 \operatorname{div} \xi\right) + \rho_0 g \frac{\partial \xi_3}{\partial y} = 0$$

$$\rho_0 \partial_t^2 \xi_3 - \frac{\partial}{\partial z}\left(\rho_0 c^2 \operatorname{div} \xi\right) - \rho_0 g \left(\frac{\partial \xi_1}{\partial x} + \frac{\partial \xi_2}{\partial y}\right) = 0$$

where we have used $\quad c^2 \dfrac{d\rho_0}{dz} = -\rho_0 g$

In general one can use

$$\xi = \tilde{\xi}(z) \exp i\left(k_1 x + k_2 y - \omega t\right)$$

substituting and simplifying one obtains an eqn. of the form:

$$\frac{d^2 \tilde{\xi}_3}{dz^2} + f(z) \frac{d\tilde{\xi}_3}{dz} + r(z)\, \tilde{\xi}_3 = 0$$

substitute $\quad \tilde{\xi}_3 = h \exp\left[-\frac{1}{2}\int_0^z f(\sigma)\, d\sigma\right]$

Then one obtains:

$$\frac{d^2 h}{dz^2} + \gamma^2 h = 0 \qquad \gamma^2 = r - \frac{1}{4} f^2 - \frac{1}{2}\frac{df}{dz}$$

The solutions are oscillatory only if $\gamma^2 > 0$

Otherwise we have an unstable solution.

To see a simplified version ignore the variations along the y direction and consider the problem in 2-d, x - z plane :

$$\rho_0 \ddot{\xi}_1 - \frac{\partial}{\partial x} \rho_0 c^2 \operatorname{div} \xi + \frac{\partial}{\partial x} \rho_0(z) g \xi_3 = 0$$

$$\operatorname{div} \xi = i k_1 \xi_1 + \frac{\partial \xi_3}{\partial z}$$

$$- \rho_0 \omega^2 \tilde{\xi}_1 - \rho_0(z) c^2(z) \, i k_1 \left(i k_1 \xi_1 + \frac{\partial \xi_3}{\partial z} \right) + \rho_0(z) g \, i k_1 \xi_3 = 0$$

$$\Rightarrow \left(- \rho_0 \omega^2 + \rho_0 c^2 k_1^2 \right) \xi_1 - i k_1 \rho_0 c^2 \frac{\partial \xi_3}{\partial z} + g \rho_0 i k_1 \xi_3 = 0$$

$$\Rightarrow \quad \xi_1 = \frac{i k_1}{\left(c^2 k_1^2 - \omega^2 \right)} \left(c^2 \frac{\partial \xi_3}{\partial z} + g \xi_3 \right)$$

From the z component :

$$\rho_0 \omega^2 \xi_3 - \frac{\partial}{\partial z} \left[\rho_0 c^2 \left(i k_1 \xi_1 + \frac{\partial \xi_3}{\partial z} \right) \right] - \rho_0 g \, i k_1 \xi_1 = 0$$

$$\Rightarrow \quad \frac{d^2 \xi_3}{d z^2} + f(z) \frac{d \xi_3}{d z} + r(z) \xi_3 = 0$$

with $\quad f(z) = \dfrac{d}{dz} \ln \left(\dfrac{\rho_0}{b^2} \right)$

$$r(z) = b^2 - \frac{k_1^2}{\omega^2} g \frac{d}{dz} \ln \left(\frac{\rho_0}{b^2} \right) - \frac{k_1^2}{\omega^2} \frac{g^2}{c^2}$$

with $\quad b^2 = \dfrac{\omega^2}{c^2} - k_1^2$

• ~~Now~~

• Note 1 : at high frequencies, $\omega \to \infty$

$r(z) = b^2$ pure acoustic modes, effects of gravity is negligible.

• consider isothermal atmosphere :

$$\Rightarrow \frac{dp_0}{dz} = -\rho_0 g \Rightarrow \frac{d\rho_0}{dz} = -\frac{\rho_0 g}{c^2} = -\frac{\rho_0}{H}$$

scale height

$$\Rightarrow \rho_0(z) = \rho_{00} \exp(-Hz)$$

$$f(z) = \frac{d}{dz}(\ln \rho), \quad r = \frac{\omega^2}{c^2} - k_1^2 + k_1^2\left(\frac{N^2}{\omega^2}\right)$$

where $N^2 = -g\left[\frac{d}{dz}\ln\rho_0 + \frac{g}{c^2}\right]$

– Brunt - Vaisalla frequency.

$$\gamma^2 = \frac{\omega^2}{c^2} - k_1^2 + \frac{k_1^2}{\omega^2}N^2 - \frac{1}{4}\left(\frac{d}{dz}\ln\rho\right)^2 - \frac{1}{2}\frac{d^2}{dz^2}\ln\rho$$

In general the dispersion relation is given by

$$\gamma = F(k_1, k_2, \omega)$$

• The waves are dispersive :

phase velocity $\frac{\omega}{k}$ is not a constant

$$v_g \equiv \text{group velocity} = \frac{\partial\omega}{\partial k}$$

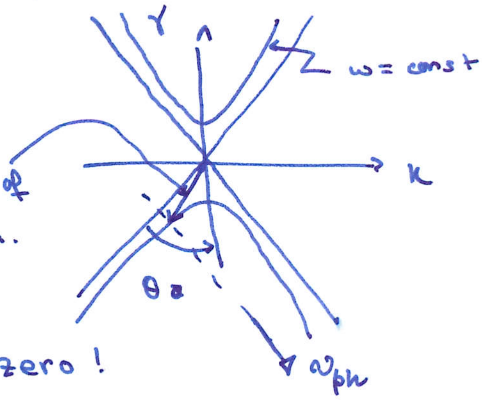
- The waves are anisotropic.

consider a surface in k_1, k_2, γ space over which ω is constant. These are called propagation surfaces.

e.g. $$\pm \frac{k^2}{a^2} \pm \frac{\gamma^2}{b^2} = 1 \qquad \text{with } k^2 = k_1^2 + k_2^2$$

These are either ellipsoids of revolution or hyperboloids of revolution.

$\theta = \tan^{-1}\left(\frac{a}{b}\right)$



- For this wave vorticity is non-zero!

- The ellipsoid case is similar to ~~wave~~ ~~for~~ propagation of electromagnetic waves in anisotropic crystals.

- The wave speed $\vec{v}_{ph} = \frac{\partial \omega}{\partial \vec{k}}$ is <u>not</u> along $\vec{k}$.

- comments on energy conservation in waves and a way of deducing the wave equation from a minimization principle.

- often a better intution about these waves can be obtained by assuming the $c(z)$ is a slow function z. slow compared to the wavelength

6.2 A flavour of Helioseismology :

Consider a spherical star. Let us ignore rotation and magnetic field. Consider a stationary state. with eqn. of state

$$p = p(\rho, T; X)$$

composition

We need not consider the exact eqn of state but assume that perturbations are adiabatic.

$$\gamma = \left.\frac{\partial \ln p}{\partial \ln \rho}\right|_s$$

and ideal gas $\quad p = \frac{R \rho T}{\mu}$

mean molecular weight.

The ~~t~~ stationary state is provided by stellar evolution models. Those are not our concern. Assume that we know them.

The basic state satisfies (assuming spherical symmetry)

$$\frac{dp_0}{dr} + g_0 \rho_0 = 0 \quad , \quad g_0 = \frac{G m_0}{r^2}$$

$$m_0(r) = 4\pi \int_0^r \rho_0(s)\, s^2\, ds$$

These would be enough for us to start.

The question is : what are the waves that solves the linearized equations about this steady state ?

Linearized equations :

$$v = D_t \xi \simeq \partial_t \xi$$

$$\rho_0 \partial_t v = -\nabla p - g_0 \hat{r} \rho + \rho_0 \nabla \Phi$$

$$\nabla^2 \Phi = -4\pi G \rho$$

$$\rho + \mathrm{div}(\rho_0 \xi) = 0$$

$$\& \quad p = c_0^2 \rho$$

This is a new term compared to the case of internal gravity waves.

ρ, p, v, Φ, are perturbations.

$\rho_0, p_0, v_0 = 0, \Phi_0$ are the stationary state.

- consider only radial pulsations :

$$\xi = (\xi, 0, 0)\, r$$

merely convention.

Proceeding in a way very similar to the internal gravity waves, we obtain :

$$r \rho \ddot{\xi} + 4 \frac{dp_0}{dr} \xi - \frac{\partial}{\partial r}\left[\gamma p_0 \left(r \frac{\partial \xi}{\partial r} + 3\xi\right)\right] = 0$$

- Lagrangian and Eulerian perturations :

For any quantity $f_0(\vec{x})$

The Eulerian variation $f(\vec{x})$

The Lagrangian variation $f(\vec{x}+\vec{\xi}) \equiv \delta f$

$$f(\vec{x}+\vec{\xi}) = f(\vec{x}) + \vec{\xi} \cdot \frac{\partial f}{\partial \vec{x}}$$

$$\Rightarrow \quad \boxed{\delta f = f + (\vec{\xi} \cdot \vec{\nabla}) f}$$

The momentum eqn. in Lagrangian frame:

$$r \rho \ddot{\xi} = -\frac{\partial \delta p}{\partial r} + 4 g \rho_0 \xi$$

The continuity eqn in Lagrangian frame:

$$\frac{\delta \rho}{\rho_0} + \frac{1}{r^2} \frac{\partial}{\partial r}(r^3 \xi) = 0$$

- ~~Boundary~~ ~~conditions~~ :

- Self - Adjoint form :

$$\mathcal{L} \xi = 0$$

where
$$\mathcal{L} \xi := \frac{d}{dr}\left(\gamma p r^4 \frac{d\xi}{dr}\right) + \left\{ r^3 \frac{d}{dr}\left[(3\gamma - 4) p\right] + r^4 \rho \omega^2 \right\} \xi = 0$$

In principle the problem of radial pulsations is now solved. We merely have to solve for the eigenfunctions of this self adjoint operator with appropriate boundary conditions.

- Boundary conditions :

- Regularity at $r = 0$.

$$\xi = r^a \sum_{k=0}^{\infty} A_k r^k$$

$r=0$ is a singular point of the equation. We can do a power-series expansion around it and eventually :

$$\frac{d\xi}{dr} = 0 \quad \text{at} \quad r = 0.$$

- Outer boundary condition can have several choices.

One choice : the corona adjusts itself instantaniously to a hydrostatic equilibrium.

$$4\pi r^2 p = g\, m_c$$

mass of the corona.

Linearize, use defn of ξ, and the continuity eqn. to obtain

$$\gamma R \frac{d\xi}{dr} + (3\gamma - 4)\xi = 0$$

- The condition at $r=0$ is a reflecting condition.

  The condition at $r=R$ is also reflecting.

  So modes can be confined in a star.

- <u>nomenclature</u> :

  The problem

  $$\mathcal{L}\xi = 0$$

  can now be solved for eigenfunctions of $\mathcal{L}$ give a ω. There are a discrete ~~set~~ set of eigenfunctions ξ_n with eigenfrequencies ω_n. The eigenfunctions can be shown to be orthogonal.

  When organized with frequency ω_n, the smallest ~~ω_n~~ frequency ω_1, for $n=1$ is called the fundamental.

- It can be shown that the frequencies have a lower limit (

$$\omega^2 \geq (3\gamma - 4)\frac{GM_\odot}{R^3}$$
$$= (3\gamma - 4)\,\omega_0^2$$

- ~~The wave equa~~

- ~~Perturb~~

- Approximate solution of the wave equation:

Uses the JWKB method:

$$\xi = \mathrm{Re}\left[A \exp i\lambda \int \psi \, dr\right] \qquad \lambda = \frac{\omega}{\omega_0}$$

A similar but more mathematically involved method applies to pulsations that are not solely radial. These are (~~misnome~~ called non-radial pulsation, which is a misnomar because they always have a radial component.

They obey a dispersion relation:

$$k^2 = \frac{\omega^2 - \omega_c^2}{c^2} - \frac{l(l+1)}{r^2}\left(1 - \frac{N^2}{\omega^2}\right)$$

with
$$\omega_c = \frac{c^2}{4\mathcal{H}^2}\left(1 - 2\mathcal{H}'\right) - \frac{g}{h}$$

$$N^2 = g\left(\frac{1}{\mathcal{H}} - \frac{g}{c^2} - \frac{2}{h}\right)$$

$$\frac{1}{\mathcal{H}} = \frac{1}{H} + \frac{1}{H_f} + \frac{1}{h} + \frac{1}{r}$$

H ↑ density scale height

$$\frac{1}{h} = \frac{1}{H_g} + \frac{2}{r}$$

H_g ↑ gravity scale height

- small ω modes are determined by N the Brunt-Vaissala frequency. These are the g modes.

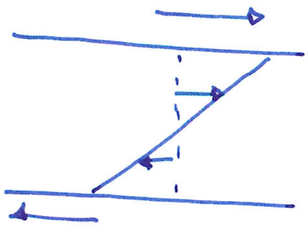
- Large ω modes are determined by sound waves dominated by pressure. These are the p modes.

- In reaching this conclusion the perturbation of the gravitational potential has been ignored.
  — cowling's approximation.

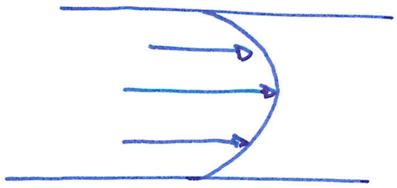
6.3 Instabilities in shear flows.

So far we have dealt with instabilities of flows for which the unperturbed state had zero velocity. The problem can become quite a bit more interesting if the ~~of~~ unperturbed state has shear: one component of velocity is a function of a different coordinate direction ~~to~~ $u_x(y)$. Such flows are very relevant.

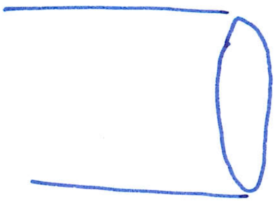
Some examples are:



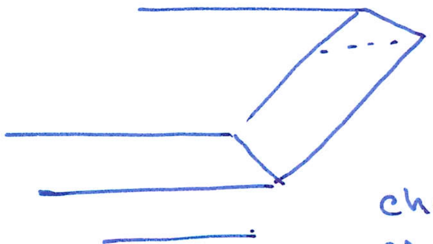
Plane Couette



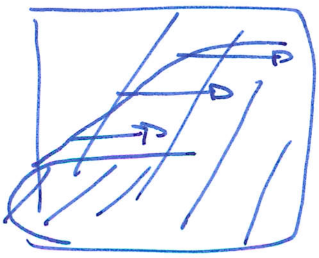
Plane Poiseuille



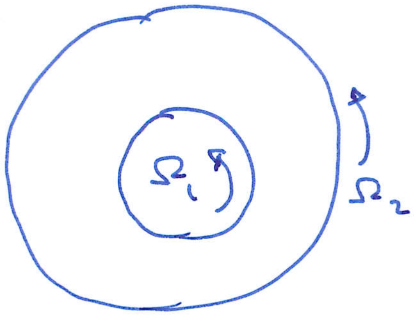
Pipe flow



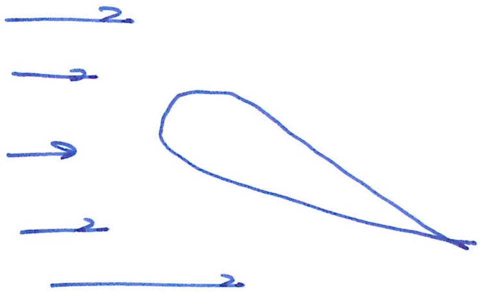
channel flow



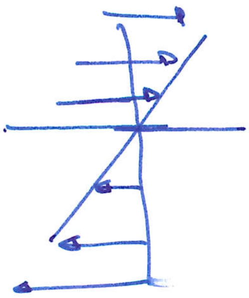
They can be considered in both viscous and inviscid formulation. Although the inviscid formulation can have fundamental problem with boundary conditions.



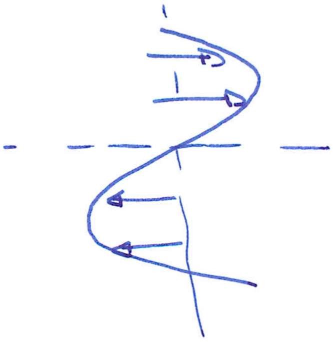
Taylor-couette flow



unbounded shear flows.



homogeneous shear flow



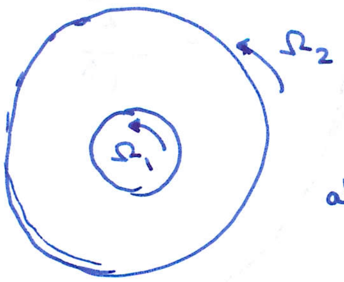
Kolmogorov flow.

And so on and so forth.

Interestingly Taylor-couette although may look quite complicated is in many ~~ma~~ ways one of the best to study through linear analysis. There is a kaleidoscope of possible behaviour. The simplest case is the following:

~~Let us consider the incompress~~

6.3.1 ~~6.4~~ Inviscid Taylor - Couette flow :



$$\Omega(r) = A + \frac{B}{r^2}$$

at $R = R_1$, $\Omega = \Omega_1$

$R = R_2$ $\Omega = \Omega_2$

$$\Rightarrow \quad A = -\Omega_1 \eta^2 \, \frac{1 - \mu/\eta^2}{1 - \eta^2} \qquad \mu = \frac{\Omega_2}{\Omega_1}$$

$$B = \Omega_1 \, \frac{R_1^2 (1 - \mu)}{1 - \eta^2} \qquad \eta = \frac{R_1}{R_2}$$

In the absence of viscosity; there ~~is~~ $\Omega(r)$ can be in general any function of r. What are the necessary and sufficient conditions for linear stability of the flow?

Answer : $\frac{d}{dr}(r^2\Omega)^2 > 0 \iff$ stability

If $(r^2\Omega)^2$ decreases with r anywhere in the domain $\Rightarrow$ instability.

$l \equiv$ angular momentum per unit mass $= r^2\Omega$

stratification of angular momentum is stable iff it increases monotonically outward.

$$\boxed{\text{Rayleigh Criterion}}$$

Argument :

consider only axisymmetric perturbations.

$$\partial_t u_\theta + u_r \frac{\partial u_\theta}{\partial r} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} = 0$$

$$\Rightarrow \quad \cancel{\frac{d}{dt}} \; D_t (r u_\theta) = 0 \qquad u_\theta = \Omega r$$

$$\Rightarrow \quad D_t (r^2 \Omega) = 0$$

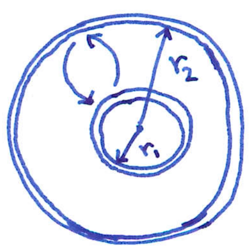
→ angular momentum is conserved.

The radial eqn.

$$\partial_t u_r + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} = \underbrace{\frac{u_\theta^2}{r}}_{(l^2/r^3)} - \frac{\partial}{\partial r}(p) \qquad \rho = 1$$

$$= -\frac{\partial}{\partial r}\left(\frac{l^2}{2r^2}\right)$$

an effective potential.



$$2\pi r_1 dr_1 = 2\pi r_2 dr_2$$

mass conservation.

The change in energy after the ~~ra~~ "exchange"

$$\left(\frac{l_2^2}{r_1^2} + \frac{l_1^2}{r_2^2}\right) - \left(\frac{l_1^2}{r_1^2} + \frac{l_2^2}{r_2^2}\right)$$

$$= (l_2^2 - l_1^2)\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) \equiv \Delta E$$

For stability this change in energy must be negative

$r_2 > r_1$, $\Rightarrow$ $\Delta E < 0$ when $l_2^2 < l_1^2$

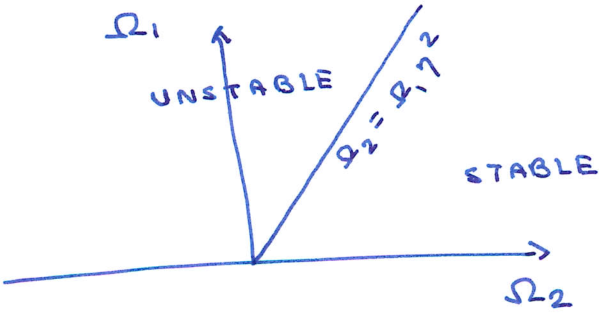
$\Rightarrow$ angular momentum should decrease outward everywhere in the domain.

• What does this imply if $\Omega(r)$ satisfies the viscous solution $\Omega(r) = A + \frac{B}{r}$?

$$\Phi(r) = \frac{1}{r^3}\frac{d}{dr}\left(r^2\Omega\right)^2$$

$$\Phi(r) > 0$$

$$\Rightarrow \quad \mu = \frac{\Omega_2}{\Omega_1} > \eta^2 \qquad \left(\eta = R_1/R_2\right)$$



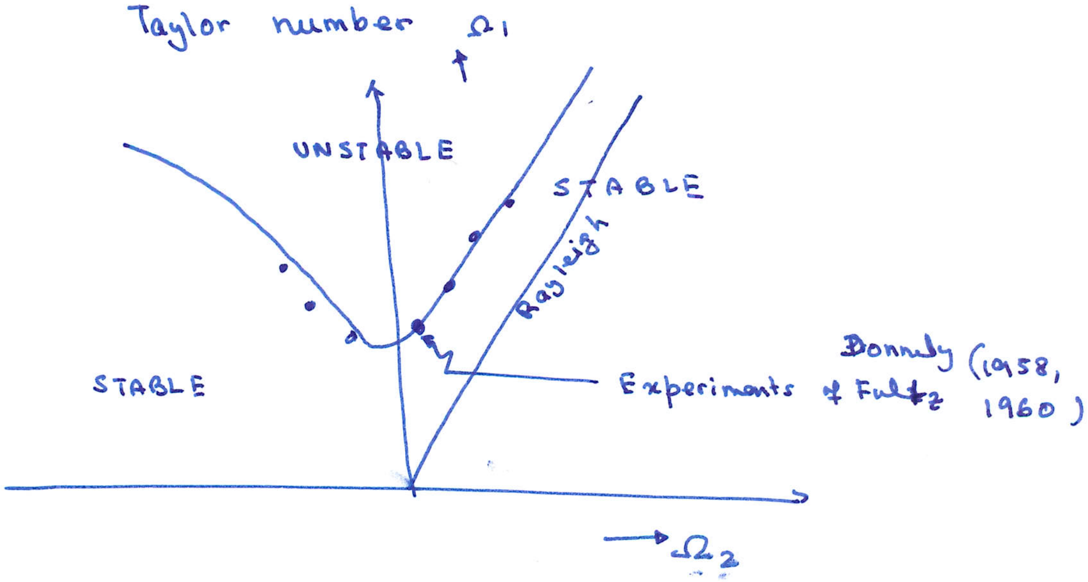
• Actual viscous calculation (starting from G I Taylor)

viscosity may postpone the onset of instability upto a critical value (<u>always?</u>)

$$T = \frac{4\Omega_1^2 R_1^4}{\nu^2}\frac{(1-\mu)(1-\mu/\eta^2)}{(1-\eta^2)^2}$$

↑

Taylor number

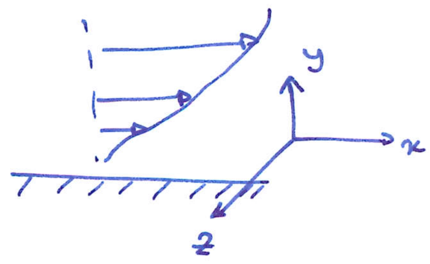


6.3.2

Orr-Sommerfeld equations

$$\mathbf{v} = \mathbf{U} + \mathbf{u}$$

$$\mathbf{U} = \hat{x}\, U(y)$$



$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{Re}\nabla^2 \mathbf{u} - \nabla p$$

$$(\mathbf{U}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{U}$$

$$U(y)\partial_x \mathbf{u} \qquad\qquad \hat{x}\, u_y \partial_y U$$

$$\eta = \omega_y, \qquad v = u_y$$

$$\left(\partial_t + U\partial_x - \frac{1}{Re}\nabla^2\right)\eta = -\partial_z v \frac{dU}{dy} \qquad \text{Squire Eqn.}$$

$$\left(\partial_t + U\partial_x - \frac{1}{Re}\nabla^2\right)\nabla^2 v = \partial_x v \frac{d^2 U}{dy^2}$$

A set of closed equations. Orr-Sommerfeld equations.

+ Boundary conditions.

$$\text{no slip} \Rightarrow \left.\begin{matrix} v = 0 \\ \omega = 0 \end{matrix}\right\} \text{ at the walls}$$

In Fourier space:
$$v = \hat{v}(y)\, e^{\lambda t}\, e^{i(\alpha x + \gamma z)}$$
$$\eta = \hat{\eta}(y)\, e^{\lambda t}\, e^{i(\alpha x + \gamma z)}$$

$$\left[\lambda + i\alpha U - \frac{1}{Re}(D^2 - k^2)\right]\hat{\eta} = -i\gamma \hat{v}\, U'$$

$$\left[\lambda + i\alpha U - \frac{1}{Re}(D^2 - k^2)\right](D^2 - k^2)\hat{v} = U'' i\alpha \hat{v}$$

$$D = \frac{d}{dy}, \qquad k^2 = \alpha^2 + \gamma^2$$

$$\left[\tilde{\lambda} + ikU - \frac{1}{Re'}(D^2 - k^2)\right](D^2 - k^2)\hat{v} - U''ik\hat{v} = 0$$

$$\tilde{\lambda} = \frac{\lambda k}{\alpha} , \quad Re' = \frac{Re\,\alpha}{k}$$

- Squire's theorem:

  A three dimensional perturbation (α, γ) at a fixed Re with growth rate λ is equivalent to a two-dimensional perturbation with wave number $(k, 0)$ but with $Re' = \frac{Re\,\alpha}{k} < Re$.

  with growth rate $\tilde{\lambda} = \frac{\lambda k}{\alpha} > \lambda$ [only real part matters]

  so for any 3D unstable mode, there exists a 2D unstable mode with a growth rate greater than the 3D one at a critical Re smaller than the 3D one. So, to find the first critical Re we need to look at only 2D perturbations.

- The linear stability problem has now been reduced to finding eigenfunctions and eigenvalues of the Orr-Sommerfeld equation. <u>But this is not a self-adjoint operator</u>.

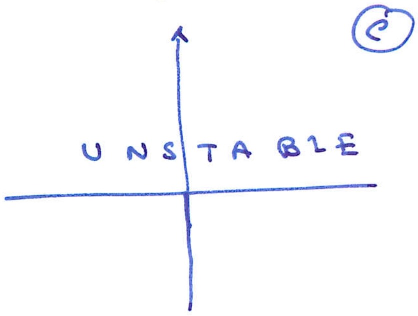
- Inviscid case: $\frac{1}{Re} = 0$

$$(U - c)(D^2 - \alpha^2) v - U'' v = 0$$

Rayleigh eqn.

with $\lambda = -i\alpha c$, $\quad c = -\frac{\lambda}{i\alpha} = \frac{i\lambda}{\alpha}$

Instability can appear when $\Im m(c) > 0$



with Boundary condition $v = 0$
(only one boundary condition)

$$\int_{y_1}^{y_2} \left[v'' - \alpha^2 v - \frac{U'' v}{U - c_1 - i c_2} = 0 \right] v^* \, dy$$

$$\Rightarrow \int_{y_1}^{y_2} \left(|v'|^2 + k^2 |v|^2 \right) dy + \int_{y_1}^{y_2} \frac{(U - c_1 + i c_2)\, U'' |v|^2}{(U - c_1)^2 + c_2^2} \, dy = 0$$

If $c_2 \neq 0$

The imaginary part of this eqn. becomes

$$i \int_{y_1}^{y_2} \frac{c_2 U'' |v|^2}{(U - c_1)^2 + c_2^2} \, dy = 0$$

$\Rightarrow$ $U''(y)$ must change sign in the domain.

$\Rightarrow$ If $U''(y)$ is non-zero in the whole domain

we ~~cannot~~ ~~have~~ must have $c_2 = 0$,

~~complete~~. the imaginary part of c must be zero.

This may look like a convincing proof of stability of inviscid flows without inflection points. But the actual matter is somewhat more delicate.

For $\mathrm{Im}(c) = 0$ we should get a neutral wave solution. It can be shown that for a plane-parallel flow the phase velocity of the ~~neutr~~ neutral wave c must be $U_{min} \leq c \leq U_{max}$

$\Rightarrow$ There will be at least one point in the domain with $U(y_0) = c$ where the Rayleigh eqn. will ~~becom~~ become singular.

What one should really do is to go back to the Orr-Sommerfeld equation and study its solutions for the limit $Re \to \infty$.

Summary of results

- Couette flow is lineary stable for all Re.
- Plane Poiseuille flow is lineary stable in the inviscid case. BUT is lineary unstable for the viscous case with an ~~ist~~ instability at $Re \sim 5772$.

  This counterintutive result was first shown by Heisenberg. (His PhD thesis)
- Pipe flow is belived to be lineary stable for all Re but not proven yet.
- In general linear stablity is a poor predictor of critical numbers where a flow becomes unstable.

Lecture VII

- Magnetorotational Instability (MRI)

~~ta In the~~

While studying the Taylor-Couette problem we already found that the stability criterion of Rayleigh is

$$\frac{1}{r^3}\frac{d\,(r^2\Omega)^2}{dr} > 0$$

What happens if there is a constant magnetic field in the vertical direction?

We lineariz the equations with the following unperturbed state $\cancel{v = (0, 0, \Omega(r)\, r)}$

$\cancel{B = 0}$,

$$v = (0,\ \Omega(r)\, r,\ 0)$$

$$B = (0,\ 0,\ B)$$

And look for perturbations that are functions of z only. Then we obtain

$$-i\omega\,\delta v_r - 2\Omega\,\delta v_\phi - \frac{ikB}{2\mu_0}\,\delta b_r = 0$$

$$-i\omega\,\delta v_\phi + \frac{\kappa^2}{2\Omega}\,\delta v_r - \frac{ikB}{2\mu_0}\,\cancel{MB}\,\delta b_\phi = 0$$

$$\cancel{-i\omega\,\delta b} \qquad -\omega\,\delta b_r = kB\,\delta v_r$$

$$-i\omega\,\delta b_\phi = \delta b_r\,\frac{d\Omega}{d\ln r} + ikB\,\delta v_\phi$$

with epicyclic frequency $\kappa^2 = \frac{1}{r^3}\frac{d}{dr}(r^2\Omega)^2$

The dispersion relation:

$$\omega^4 - \omega^2\left[\kappa^2 + 2(k\,c_A)^2\right] + (k\,c_A)^2\left[(k\,c_A)^2 + \frac{d\Omega^2}{d\ln r}\right] = 0$$

where $c_A \equiv$ Alfven velocity $= \frac{B}{\sqrt{\mu_0 \rho_0}}$

~~can become negative when~~

stability criterion:

$$(k\,c_A)^2 > -\frac{d\Omega^2}{d\ln r}$$

The problem can always become unstable unless

$$\frac{d\Omega^2}{d\ln r} > 0$$

so the Rayleigh criterion

$$\frac{d}{dr}(r^2\Omega)^2 > 0$$

should be replaced by

$$\frac{d\,\Omega^2}{dr} > 0$$

- angular momentum must be replaced by angular velocity.

- The maximum growth rate of the instability:

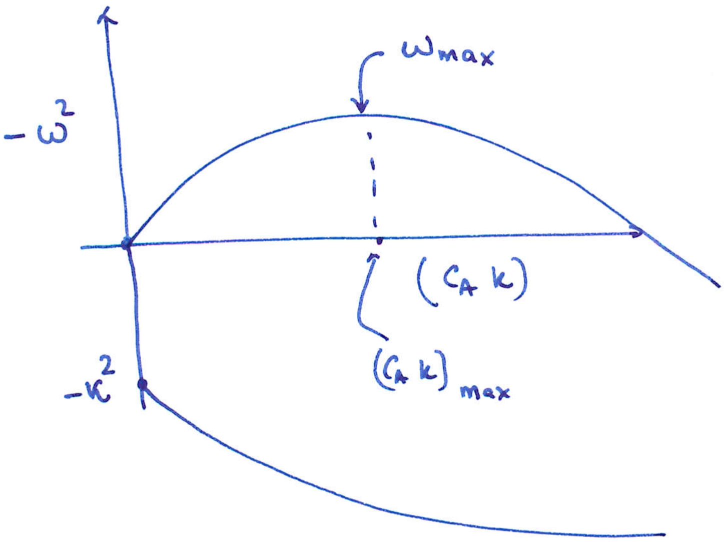
$$|\omega_{max}| = \frac{1}{2}\left|\frac{d\Omega}{d\ln r}\right|$$

when
$$(k\,c_A)^2_{max} = -\left(\frac{1}{4} + \frac{\kappa^2}{16\Omega^2}\right)\frac{d\Omega^2}{d\ln r}$$

- Keplerian rotation profile:

$$\Omega^2 r = \frac{GM_\odot}{r^2} \Rightarrow \Omega = \Omega_0 r^{-3/2}$$

$$\omega_{max} = \frac{3}{4}\Omega \qquad (c_A \cdot k)_{max} = \sqrt{\frac{15}{4}}\,\Omega$$



$$k_{max} \sim \frac{1}{c_A}$$

smaller the magnetic field larger k_{max} will be.

- It is the slow magnetosonic wave that becomes unstable.

This is the essence of the magnetorotational ~~ist~~ instability.

- Is the magnetic field ever too ~~smatly~~ small to be dynamically ignored?

- A turbulent disk:

- why do we need the MRI? Is not a Keplarian disk hydrodynamically unstable?

Rayleigh criterion implies, $\frac{d}{dr}(r^2\Omega)^2 > 0$

$$\Omega \sim r^{-3/2}, \qquad r^2\Omega \sim \sqrt{r}$$

$$\frac{d}{dr}(r^2\Omega)^2 = \frac{d}{dr} r = 0.$$

Numerical simulations show that a Keplarian disk is ~~li~~ nonlinearly stable.

Although a shear flow in general is not. !

- The difference could be from boundary conditions.

- steady state Keplerian disk (ideal)

$$\partial_t \rho + \mathrm{div}(\rho v) = 0$$

$$\partial_t (\rho v) + \mathrm{div}\left(\rho v_i v_j + p\delta_{ij}\right) = J \times B + \nabla \Phi$$

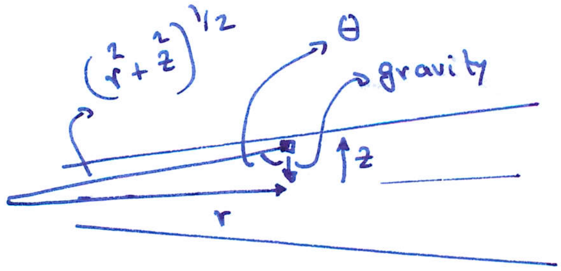
Gravity.

$$v_\phi = v_\phi + \Omega(r)\, r$$

$$c^2 = \gamma p/\rho$$

$$\partial_t B = \nabla \times (v \times B)$$

$$\Omega^2 = \frac{GM_\odot}{R^3}$$



The vertical structure of the disk

$$\frac{\partial p}{\partial z} = - \frac{GM_\odot}{(r^2+z^2)} \cos\theta \, \rho(z)$$

$$= - \frac{GM}{(r^2+z^2)} \frac{z}{(r^2+z^2)^{1/2}} \rho(z)$$

$$\simeq - \frac{GM}{r^3} z \rho = - \rho \Omega^2 z$$

$$\Rightarrow \quad \rho = \rho_0 \exp\left(-\frac{z^2}{H^2}\right) \qquad H = \frac{\sqrt{2}\, c_s}{\Omega}$$

For large r, $\Omega r > c_s$ is possible; actually quite common in astrophysical applications.

Then $\frac{H}{r} = \frac{\sqrt{2}\, c_s}{\Omega r} < 1$ thin disk approximation.

- Equation for angular momentum

$$\partial_t (\rho r v_\phi) + \nabla \cdot \Big[\rho r v_\phi v - \frac{r B_\phi}{2\mu_0} \left(\cancel{B_\phi^2} + \cancel{B_z^2} \right) (\hat{r} B_r + \hat{\phi} B_\phi)$$

$$+ \; r \left(p + \frac{B_r^2 + B_z^2}{\mu_0} \right) \hat{e}_\phi \Big]$$

$$- \nabla \cdot \Big[\qquad \Big] = 0$$

dissipative terms

On averaging

$$\partial_t \langle l \rangle + \nabla \cdot [\langle \quad \rangle] = 0$$

flux

$u = v - \Omega r$

fluctuating part

The radial component of the flux

$$W_{r\phi} = \left\langle r \left[\rho \, u_r (\Omega r + u_\phi) - \frac{B_r B_\phi}{\mu_0} \right] \right\rangle$$

$$= \Sigma \, r \, [\, r \Omega \langle u_r \rangle_\rho + \underbrace{\langle u_r u_\phi - c_{Ar} c_{A\phi} \rangle_\rho}_{\text{turbulent transport.}} \,]$$

where $\Sigma = \int_{-\infty}^{+\infty} \rho \, dz$ $\qquad c_A = \frac{B}{\sqrt{\mu_0 \rho_0}}$

$$\langle X \rangle_\rho = \frac{1}{2\pi \Sigma \Delta R \, r} \int_{\substack{r \in \Delta R \\ \phi \in 2\pi}} X \rho \, d\phi \, dr \, dz$$

If there is a non-vanishing angular momentum flux outward, then matter slowly looses angular momentum and ~~spi~~ accrete inward. So the disk looses mass at the following rate:

$$\dot{M} = - 2\pi r \Sigma \langle u_r \rangle_\rho$$

It can be shown that there is a flux of energy inward, contributing to the luminosity of the disk. Typical boundary conditions are stress-free at the inner boundary.

The Shakura Sunayev model

$$W_{r\phi} \sim \alpha c_s^2$$

- Can the MRI produce sustained turbulence? How does it saturate?
- Can we calculate α from first principles?

How does instabilities saturate?

Consider the problem in an abstract manner.

There is a critical number, the Reynold's number, the Taylor number, or the Richardson number, which when exceeds a critical value an ~~is~~ instability develops. This instability is typically of the following form:

$$u(x,t) \sim A(t)\, f(x)$$

whee $f(x)$ is an appropriate eigenfunction that satisfies the boundary conditions. $A(t)$ is typically complex. In the simplest case it obeys the following equation:

$$A(t) = e^{-i\omega t} = e^{\gamma t - i\omega_1 t} \qquad \frac{dA}{dt} = (\gamma - i\omega_1)\, e^{\gamma t - i\omega_1 t}$$

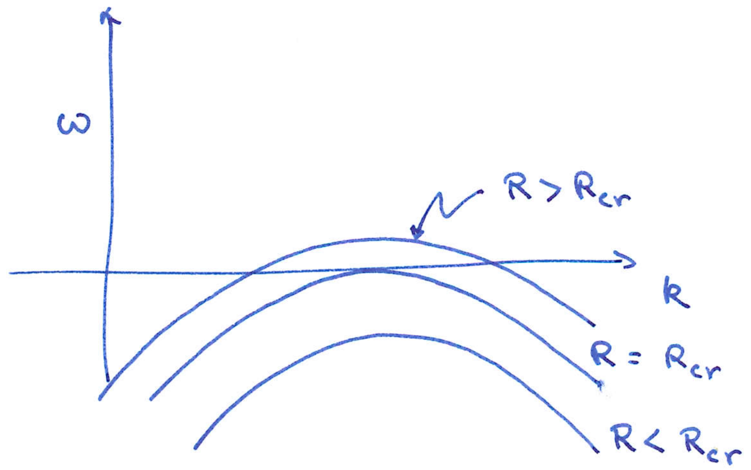
$$\frac{d}{dt}(A A^*) = A^* \frac{dA}{dt} + A \frac{dA^*}{dt}$$

$$= e^{\gamma t} \cancel{e^{i\omega_1 t}} (\gamma - i\omega_1)\, e^{\gamma t - \cancel{i\omega_1 t}} + c.c.$$

$$= 2\gamma (A A^*)$$

This is merely a restatement of the fact that the amplitude of the eigenfunctions are complex and their magnitude grows exponentially.

Now consider the problem near the critical point



Near $R = R_{cr}$ there are only a few (maybe even one) unstable mode that grows expontically with time with a rate γ and oscillates with a frequency ω_1. So near this point we can ignore all the other possible modes. So ~~t~~ our dynamical equation becomes

$$\frac{d|A|^2}{dt} = 2\gamma |A|^2 + \left(\text{functions of } |A|^2 \text{ that makes } |A|^2 \text{ saturate} \right)$$

Next possible term:

$$A^2 A^* \quad , \quad (A^*)^2 A$$

* They are not allowed by two arguments:

(i) they contain terms $e^{-i\omega_1 t}$ which when averaged over time scale longer than $1/\omega_1$ becomes zero.

(ii) they are not real.

So the first non-zero term is

$$\boxed{\frac{d}{dt}|A|^2 = 2\gamma |A|^2 \mp \mu |A|^4}$$

First example of an amplitude equation.

- Consequences:

(a) The instability saturates when $\frac{d}{dt}|A|^2 = 0$

$$\Rightarrow \quad 2\gamma |A|^2 = + \mu |A|^4$$

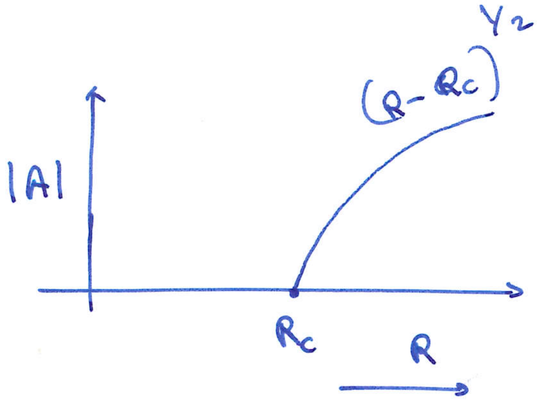
$$\Rightarrow \quad |A|^2 = \cancel{2\gamma} \left(\frac{2\gamma}{\mu}\right)$$

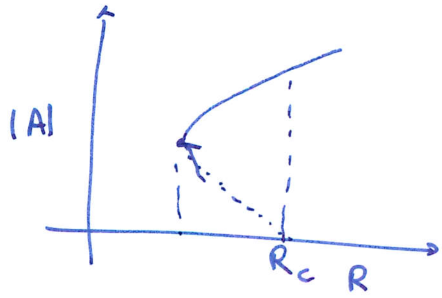
Note that γ is also a function of $R - R_c$ and goes to zero at $R = R_c$.

$$\Rightarrow \quad \gamma \sim (R - R_c) \quad \text{for } R - R_c \text{ small}$$

$$\Rightarrow \quad |A|^2 \sim R - R_c$$

$$\boxed{|A| \sim (R - R_c)^{1/2}}$$





very similar to Landau's theory of phase transition.

- If $\mu < 0$ Then $|A|$ will grow fast and very soon the amplitude eqn will not remain applicable.

But we can still apply the eqn. to study fluctuations below $R = R_c$.

Then γ is negative; but

$$\frac{d|A|^2}{dt} = 2\gamma\, |A|^2 - \mu\, |A|^4$$

can become positive for large enough $|A|$.

$\Rightarrow$ The motion will become unstable even for $R < R_{cr}$ but not for infinitismal ~~ampl~~ perturbations but for finite perturbations for which

$$\mu\, |A|^4 > 2\gamma\, |A|^2$$

$$\Rightarrow \quad |A|^2 > \frac{2\gamma}{\mu}$$

$$\Rightarrow \quad A > \left(\frac{2\gamma}{\mu}\right)^{1/2}$$

- The above arguments apply only when the instability selects a few unstable mode. If there are whole ranges of unstable modes then we ~~im~~ immediately land up in a more complicated situation.

- Furthermore, e.g for shear flow ~~turb~~ instabilities, it is not clear how this mechanism may work, as the linear theory does not give any instability in many cases.

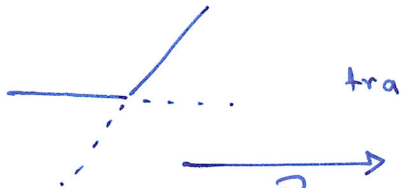
- γ can be calculated from linearized equations but not μ. How do we obtain μ from the dynamical equations?. We have to average our equations of ω_i and write an effective theory. This has been done in certain cases by using the method of ~~small~~ multiple scales.

Bifurcations

- $\partial_t A = R - A^2$

saddle node

- $\partial_t A = RA - A^2$



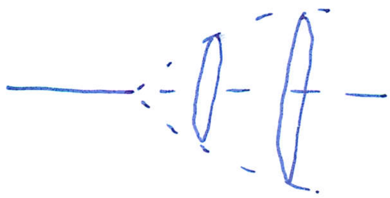
transcritical.

- $\partial_t A = RA - g A^3$

pitchfork

super critical subcritical.

- $$\partial_t A_1 = -A_2 + RA_1 - (A_1^2 + A_2^2) A_1$$
$$\partial_t A_2 = -A_1 + RA_2 - (A_1^2 + A_2^2) A_2$$



Hopf bifercation.

b Nonlinear stability

If ~~the~~ a point in the neighbourhood of a fixed point remains close to it ~~to~~ for all times then ~~the point is stable~~. fixed point in nonlinearly stable. If the point asymtotically tends to the fixed point then it is asymptotically stable.

Lecture VII

Turbulence :

We shall essentially study homogeneous and isotropic turbulence. What is this strange beast?

- Demonstration of flow behind a circular cylinder at ~~vari~~ various Reynold's number from ~~the old~~ The Album of Fluid Motion (TAFM)

TAFM	Fig 1
TAFM	Fig 24
TAFM	Fig 40
TAFM	Fig 41, 42 43, 44
TAFM	Fig 152 Grid Turbulence , 153

Note the gradual loss of symmetry and statistical restoration of symmetry at very high Reynold's ~~nu~~ number.

- Navier-stokes with periodic boundary conditions

$$\partial_t u + (u \cdot \nabla) u = \nu \nabla^2 u - \nabla p + f$$

external force to reach a statistically stationary state.

- ~~Probabilistic~~ de

- Our problem now requires a statistical description.

  We give up on writing down a solution given the initial condition and the force. We study only statistical quantities.

Assumptions

- ~~For a given f, we are limited to large length scales we would to obtain f.~~

- An invariant measure exists.

Conservation laws

$$\langle f \rangle = \frac{1}{L^3} \int_V f(\vec{x})\, d\vec{x}$$

For periodic functions:

$$\langle \partial_j f \rangle = 0$$

$$\langle (\partial_j f)\, g \rangle = - \langle f\, \partial_j g \rangle$$

$$\langle (\nabla^2 f)\, g \rangle = - \langle (\partial_i f)(\partial_i g) \rangle$$

$$\langle u \cdot (\nabla \times v) \rangle = \langle (\nabla \times u) \cdot v \rangle$$

$$\langle u \cdot \nabla^2 v \rangle = - \langle (\nabla \times u) \cdot (\nabla \times v) \rangle \quad \text{if } \nabla \cdot v = 0$$

- conservation of momentum

$$\frac{d}{dt} \langle v \rangle = 0$$

- conservation of energy

$$\frac{d}{dt} \left\langle \frac{1}{2} v^2 \right\rangle = -\frac{1}{2} \nu \sum_{i,j} \left(\partial_i v_j + \partial_j v_i \right)^2$$

$$= - \nu \langle |\omega|^2 \rangle$$

- conservation of helicity

$$\cancel{\frac{d}{dt} \langle \omega \cdot u \rangle} =$$

$$\frac{d}{dt} \left\langle \frac{\omega \cdot v}{2} \right\rangle = - \nu \langle \omega \cdot (\nabla \times \omega) \rangle$$

mean enstrophy

$$E \equiv \frac{1}{2}\langle v^2\rangle, \quad \Omega = \frac{1}{2}\langle \omega^2\rangle$$

$$H \equiv \frac{1}{2}\langle v\cdot\omega\rangle \qquad H_\omega \equiv \frac{1}{2}\langle \omega\cdot(\nabla\times\omega)\rangle$$

$$\frac{d}{dt}E = -2\nu\Omega \qquad \frac{dH}{dt} = -2\nu H_\omega$$

$\varepsilon = -\frac{d}{dt}E$ is the mean energy dissipation per unit mass.

$$\frac{d\Omega}{dt} = -2\nu P \quad \text{where} \quad P \equiv \langle \tfrac{1}{2}(\nabla\times\omega)^2\rangle$$

→ Palinstrophy (mean)

Probilistic tools and ideas:

- A random variable.
- probability distribution function

$$\begin{pmatrix}\text{Probability that } x \\ \text{lies between } X \text{ to } X+dX\end{pmatrix} = P(X)\,dX$$

$$\Rightarrow \quad \int P(x)\,dx = 1 \qquad \text{normalization.}$$

$P(x)$ is everywhere positive

- cumulative probability:

$$Q(X) = \int_{-\infty}^{X} P(x)\,dx$$

$$P(X) = \frac{dQ}{dX}$$

- moments

$$\cancel{\langle x^m \rangle} = \qquad \cancel{p(}$$

$$\cancel{\langle v^2 \rangle =}$$

$$\langle v^m \rangle = \int x^m \, P(x) \, dx$$

may not always exist

$$\sigma^2 = \langle v^2 \rangle \quad \text{variance}$$

$$S = \frac{\langle v^3 \rangle}{\left(\langle v^2 \rangle\right)^{3/2}} \quad \text{skewness}$$

$$F = \frac{\langle v^4 \rangle}{\left(\langle v^2 \rangle\right)^{2}} \quad \text{Flatness}$$

- characteristic function

$$K(z) = \langle e^{izv} \rangle = \int e^{izx} \, P(x) \, dx$$

Characteristic function of a sum of two independent random variables is the product of their individual characteristic functions.

$$p(x) = \int K(z) \, e^{-izx} \, dz$$

They are Fourier transforms of each other

- centered random variable: $\langle v \rangle = 0$

- Gaussian random variable:

$$K(z) = \langle e^{izv} \rangle = e^{-\frac{1}{2}\sigma^2 z^2}$$

$$p(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-x^2/2\sigma^2}$$

- Multidimensional Gaussian:

$$\Gamma_{ij} \equiv \langle v_i v_j \rangle$$

$$K(z) \equiv \left\langle \exp i(\vec{z} \cdot \vec{v}) \right\rangle$$

$$= \cancel{\exp -\tfrac{1}{2} z_j z_k \Gamma_{jk}} \rangle$$

$$= \exp\left(-\tfrac{1}{2} z_j z_k \Gamma_{jk}\right)$$

- Gaussian integration by parts:

If v is a Gaussian random variable

$$\langle v f(v) \rangle = \langle v^2 \rangle \left\langle \frac{\partial f}{\partial v} \right\rangle$$

~~proof~~ $$\int_{-\infty}^{+\infty} v f(v) \frac{e^{-v^2/2\sigma^2}}{(2\pi\sigma^2)^{1/2}} dv$$

$$= \cancel{f(v) \int_{-\infty}^{+\infty} \frac{v\, e^{-v^2/2\sigma^2}}{(\ )} dv} \Big|_{-\infty}^{+\infty}$$

$$\cancel{- \int \left(\frac{\partial f}{\partial v}\right) \int v\, e^{-v^2/2\sigma^2} dv}\; dv$$

prove by integration by parts.

• Generalization to multidimensional Gaussian variables:

$$\langle v_j f(v) \rangle = \Gamma_{ji} \left\langle \frac{\partial f}{\partial v_i} \right\rangle$$

• To Gaussian random functions

$$\langle u(x) \mathcal{F}[u] \rangle = \langle u(x) u(x') \rangle \left\langle \frac{\delta \mathcal{F}}{\delta u(x')} \right\rangle$$

A very useful theorem as we shall see soon.

• Moments of Gaussian random variable:

$$\langle v \rangle = 0$$

$$\langle v^{2m+1} \rangle = 0$$

$$\langle v^{2m} \rangle = \int x^{2m} \frac{e^{-x^2/2\sigma^2}}{(\ )} dx$$

$$= \begin{pmatrix} \text{combinatorial} \\ \text{factor} \end{pmatrix} \left[\langle v^2 \rangle \right]^m$$

<u>proof</u>

clearly it holds for $m = 1$.

~~For~~ Assume it holds for $m - 1$

$$\langle \underbrace{v^2 . v^2 ... v^2}_{2(m-1)} \rangle = (\ ) \underbrace{\langle v^2 \rangle \langle v^2 \rangle \cdots \langle v^2 \rangle}_{m-1}$$

Now $$\langle \underbrace{v^2 . v^2 ... v^2}_{2m} \rangle = (\,) \langle v^2 \rangle \left\langle \frac{\partial}{\partial v} v^{2m-1} \right\rangle$$

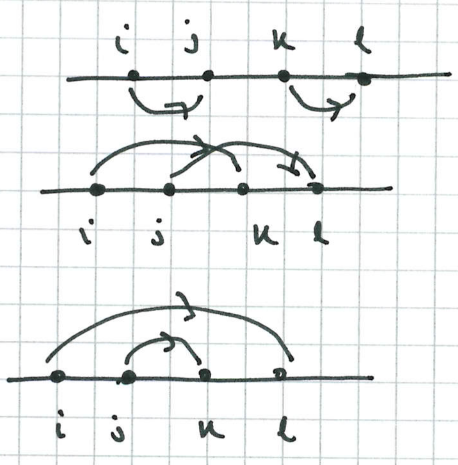
$$\langle v \quad v^{2m-1} \rangle$$

$$= (\ ) \langle v^2 \rangle \langle v^{2m-2} \rangle (2m-1)$$

$$\Rightarrow \quad \langle v^{2m} \rangle = \underbrace{\left[(2m-1) \cdots 1 \right]}_{\text{combinatorial factor.}} \underbrace{\langle v^2 \rangle \cdots \langle v^2 \rangle}_{m}$$

Now for vector valued Gaussians:

$$\langle v_i v_j v_k v_l \rangle = \Gamma_{ij} \Gamma_{kl} + \Gamma_{ik} \Gamma_{jl} + \Gamma_{il} \Gamma_{jk}$$



– Wick's theorem.

• Random functions :

A random variable that is function of space or time or both. When it is a function of time it is often called a stochastic process.

$$\Gamma_{ij}(x, x', t, t') = \langle v_i(x,t)\, v_j(x',t') \rangle$$

is called the correlation function.

- characteristic functional

$$K[z] = \left\langle \exp\left[i \int dt \; \underbrace{z(t)}_{\text{non-random test function}} v(t) \right] \right\rangle$$

A ~~raon~~ random function is called Gaussian ~~when~~ if for all test functions $z(t)$

$$\int z(t)\, v(t)\, dt$$

is a ~~g~~ Gaussian random variable.

$$K[z] = \exp - \frac{1}{2} \int dt\, dt' \left[z(t)\, z(t')\, \Gamma(t,t') \right]$$

- Statistical symmetry.

A random function is said to be stationary (time translation invariant) if

$$v(t+h) \overset{law}{=} v(t)$$

which implies that all statistical properties (including pdf and moments) of $v(t+h)$ are same as $v(t)$.

consequently

$$\Gamma(t,t') \equiv \langle v(t)\, v(t') \rangle$$
$$= \Gamma(t-t')$$

- spectrum of stationary random functions

$$v(t) = \int e^{i\omega t} \hat{v} \, d\omega$$

$$v_F^{<}(t) = \int_{|\omega|<F} e^{i\omega t} \hat{v}(\omega) \, d\omega \qquad F \geq 0$$

low-pass filtered

$$\hat{v}(\omega) = \int e^{-i\omega t} v(t) \, dt$$

The energy in Fourier space

$$E(\omega) \equiv \langle \hat{v}(\omega) \hat{v}(-\omega) \rangle = \int e^{i\omega s} \Gamma(s) \, ds$$

for stationary random functions.

— Weiner Khinchin formula.

Kolmogorov’s theory of turbulence

Dhrubaditya Mitra (NORDITA, Stocholm)

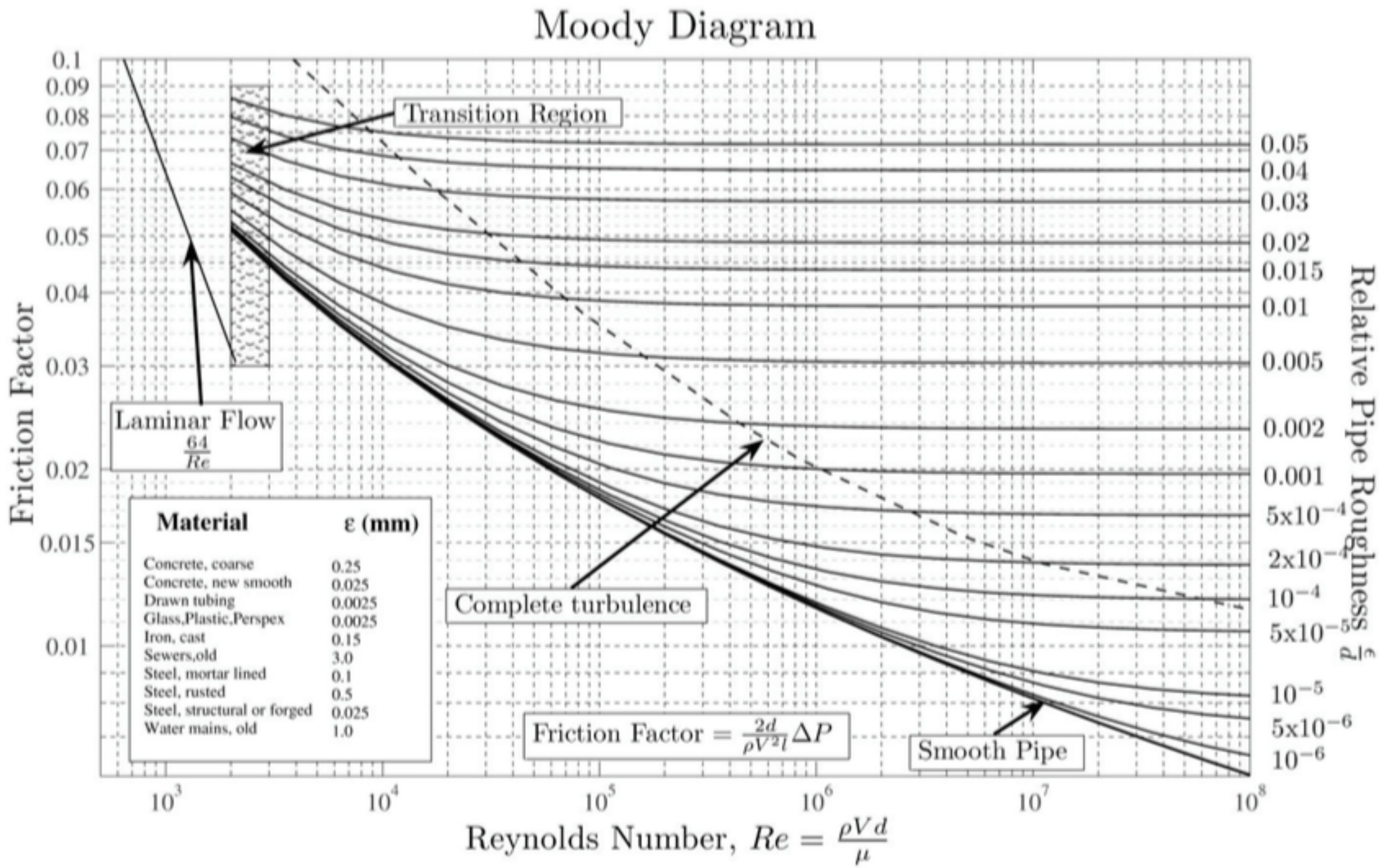
The energy dissipation law

$$\lim_{\nu \to 0} \varepsilon \equiv \lim_{\nu \to 0} \nu \langle \omega^2 \rangle \to \text{constant}$$

- In the limit of vanishing viscosity, or infinite Reynolds number the mean energy dissipation rate becomes a constant.
- Vorticity develops finer and finer structures as viscosity goes to zero.

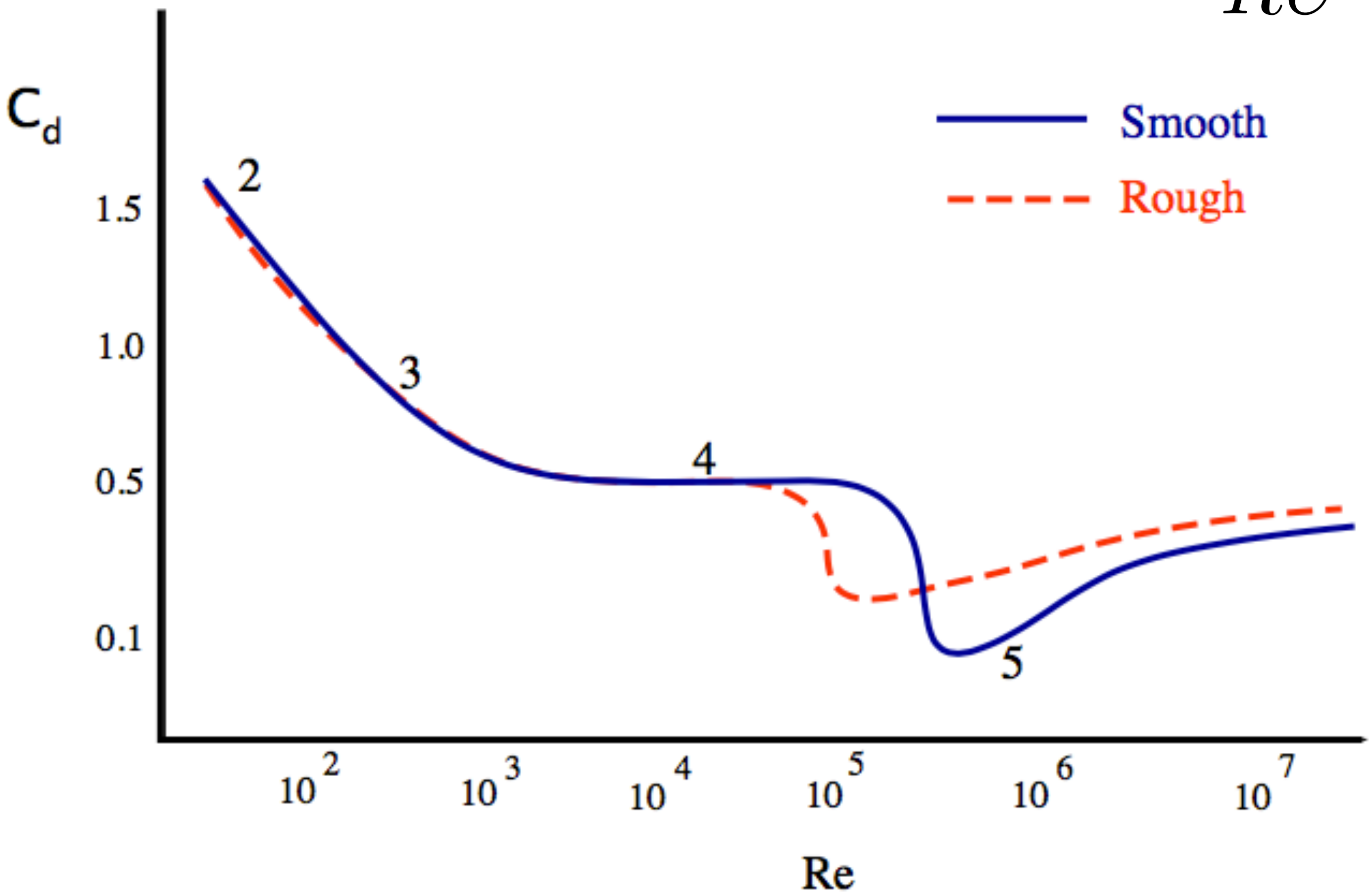
Friction factor for pipes

$$\Delta P = f_{\mathrm{D}} \frac{\rho U^2}{2} \frac{L}{D}$$



Drag law for smooth spheres

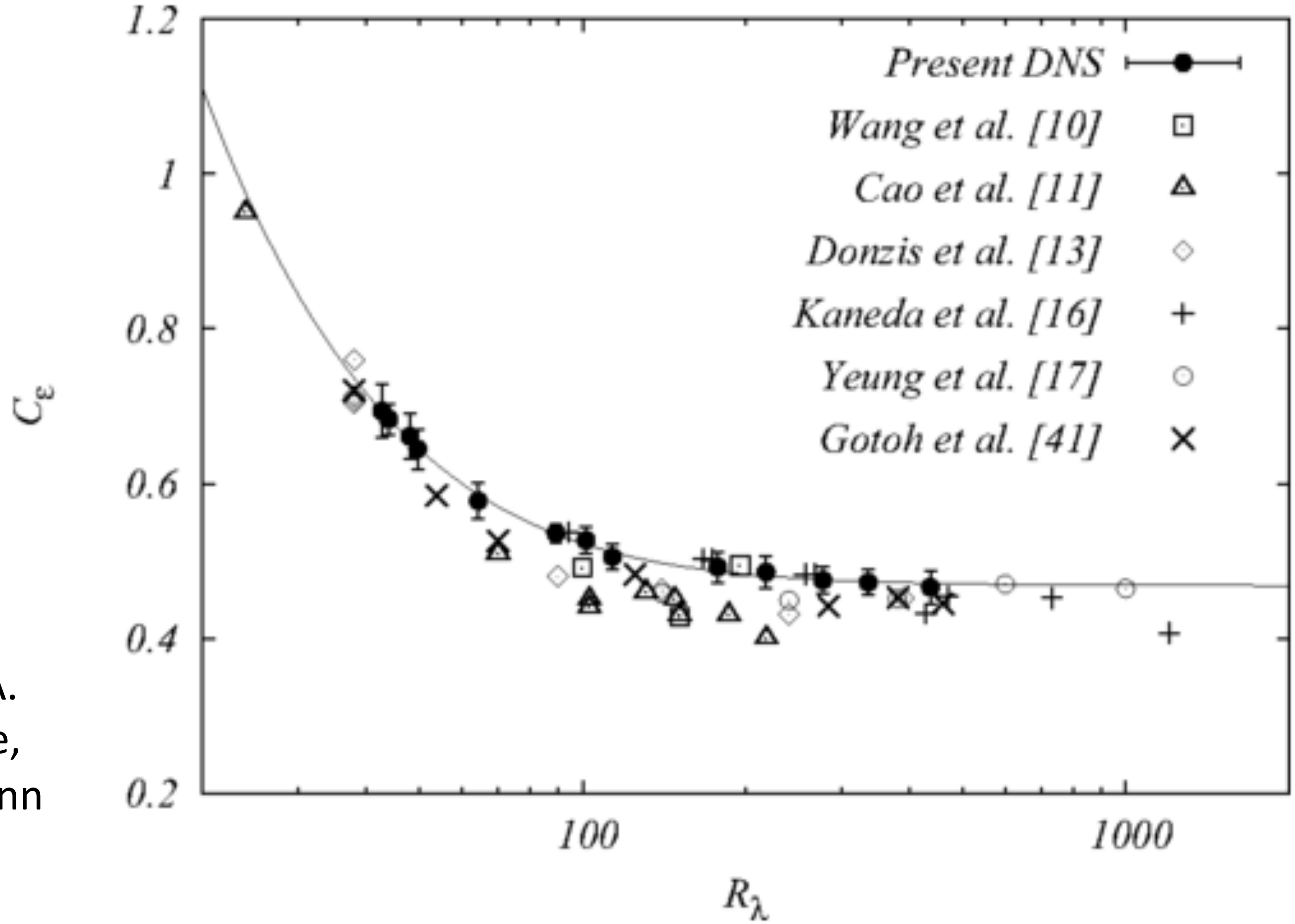
$$F_{\mathrm{drag}} \equiv C_{\mathrm{D}} \rho A U^2 \qquad C_{\mathrm{D}}^{\mathrm{Stokes}} \equiv 6\frac{\nu}{RU} \equiv \frac{6}{\mathrm{Re}}$$



By NASA - http://www.grc.nasa.gov/WWW/k-12/airplane/dragsphere.html

Simulations of HIT

$$\varepsilon \equiv C_\varepsilon \frac{U^3}{L}$$



W. D. McComb, A. Berera, S. R. Yoffe, and M. F. Linkmann Phys. Rev. E **91**, 043013, 2015

Kolmogorov's theory

- Correlation function and Structure functions.
- Inertial range.
- Dimensional argument.

$$\delta v(\ell) \sim (\varepsilon \ell)^{1/3}$$

- This implies that energy spectrum as a five-third law (After shell averaging)

$$E(k) \sim k^{-5/3}$$

Kolmogorov's theory

$$\partial_t u_\alpha + (u_\beta \partial_\beta) u_\alpha = \nu \partial_{\beta\beta} u_\alpha - \partial_\alpha p$$

$$\partial_\beta u_\beta = 0$$

It is useful to think in Fourier space:

$$u_\alpha(x) = \int \hat{u}_\alpha(k)\, e^{ik\cdot x}\, dk$$

$$\partial_\beta u_\alpha(x) = \int i k_\beta \hat{u}_\alpha(k)\, e^{ik\cdot x}\, dk$$

$$u_\beta \partial_\beta u_\alpha = \partial_\beta (u_\alpha u_\beta)$$

$$u_\alpha(x) = \int e^{ipx}\, \hat{u}(p)\, dp$$

$$u_\beta(x) = \int e^{iqx}\, \hat{u}(q)\, dq$$

$$\widehat{u_\beta \partial_\beta u_\alpha} = \int e^{-ikx}\, \partial_\beta (u_\alpha u_\beta)\, dx$$

Let us worry about the ∂_β later but look at the product first:

$$\widehat{u_\alpha u_\beta}(k) = \int u_\alpha(x)\, u_\beta(x)\, e^{-ikx}\, dx$$

$$= \int \hat{u}_\alpha(p)\, \hat{u}_\beta(q)\, e^{-i(k-p-q)}\, dx\, dp\, dq$$

$$= \int u_\alpha(p)\, u_\beta(q)\, dp\, dq\, \delta(k-p-q)$$

$$\widehat{\partial_\beta(u_\alpha u_\beta)} = i k_\beta \int u_\alpha(p)\, u_\beta(q)\, dp\, dq\; \delta(k-p-q)$$

Incompressibility can be imposed by a projection operator.

$$\cancel{P_{ij}}\quad P_{\alpha\beta}(k) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}$$

consider a vector function $u_\alpha(k)$, let $\cancel{v_\beta(k) =}$

$$v_\alpha(k) = P_{\alpha\beta}(k)\, u_\beta(k)$$

Then $\cancel{k_\alpha v_\alpha}$

$$k_\alpha v_\alpha(k) = k_\alpha P_{\alpha\beta}(k)\, u_\beta(k)$$

$$= k_\alpha\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) u_\beta(k)$$

$$= \left(k_\alpha - k_\alpha \frac{k_\alpha k_\beta}{k^2}\right) u_\beta(k) = 0$$

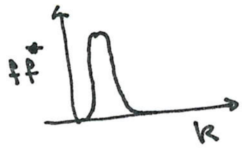
so the incompressible Navier-Stokes eqn. in Fourier space:

$$\boxed{\partial_t \hat{u}_\alpha(k) = P_{\alpha\beta}(k)\, i k_\gamma \int \hat{u}_\beta(p)\, \hat{u}_\gamma(q)\, \delta(p+q-k)\, dp\, dq - \nu k_\beta k_\beta\, \hat{u}_\alpha(k) + \hat{f}(k)}$$

peaks at high k

limited to small k.

There exists a range of scales (or Fourier modes) where the dynamics is dominated by the Non-linear term.



This is neither very small k, nor very large k, but intermediate k. This is called the inertial range.

Energy flux :

$$U_{ij}(k,t) \equiv \langle \hat{u}_i \hat{u}_j \rangle$$

averaging over the statistically stationary state of turbulence.

- comments on equilibrium, near-equilibrium and non-equilibrium. Non-equilibrium stationary state and general presence of flux. Analogies with heat conduct

$$\cancel{\partial_t U_{\alpha\alpha}(k,t)} =$$

$$\partial_t U_{\alpha\beta}(k,t) = \langle \hat{u}_\alpha \partial_t \hat{u}_\beta \rangle + \langle \hat{u}_\beta \partial_t \hat{u}_\alpha \rangle$$

Ultimately one obtains :

$$(\partial_t + 2\nu k^2)\, E(\vec{k},t) = - P_{\alpha\beta\gamma}(k) \int_{k+p+q=0} \mathrm{Im}\left[\langle \hat{u}_\alpha(k)\, u_\beta(p)\, u_\gamma(q) \rangle\right]$$

where $$P_{\alpha\beta\gamma}(k) = k_\alpha P_{\beta\gamma}(k) + k_\beta P_{\alpha\gamma}(k)$$

The proof is left ~~an~~ as an exercise.

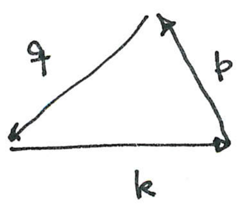
Hint : Use the dynamical equations, ~~than~~ but for the quantity $U_{\alpha\beta}(k,k') = \langle \hat{u}_\alpha(k)\, \hat{u}_\beta(k') \rangle$. Then integrate over k' and take the trace.

Remember that $E(k,t) = \langle \hat{u}(k)\, \hat{u}^*(k) \rangle$

and $\hat{u}^*(k) = \hat{u}(-k)$; because $u(x)$ is real.

Triads of interaction

$$s(k,p,q) + s(k,q,p) + s(p,k,q) = 0$$



Each such triad conserves energy.

Energy conservation scale by scale:

$$\left(\partial_t + 2\nu k^2\right) E = \int s(k,p,q)\, \delta(k+p+q)\, dp\, dq$$

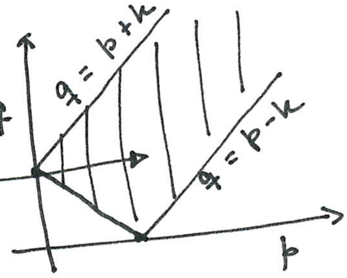
$$s(k,p,q) = s(k,q,p)$$

$$E(k,t) = E(\vec{k},t)\, 4\pi k^2$$

$$\Rightarrow \quad \left(\partial_t + 2\nu k^2\right) E = T(k,t)$$

with $\displaystyle T(k,t) = \int_{k+p+q=0} 2\pi k^2\, s(k,p,q)\, dp\, dq$

energy transfer



domain of integration.

~~The flux of energy through wave number k due to the non-linear term~~

$\cancel{\partial_t \Pi(k,t) = T}$

The energy contained in scales upto K

$$\mathcal{E}_K = \int_0^K E(k)\, dk$$

$$\partial_t \mathcal{E}_K = \int_0^K \partial_t E(k)\, dk$$

$$= -2\nu \int_0^K k^2 E(k)\, dk + \int_0^K T(k)\, dk$$

For a fixed K as $\nu \to 0$, $\nu \int_0^K k^2 E(k)\,dk \to 0$

~~For~~

~~As~~ Proof :

$$\int_0^K k^2 E(k)\,dk < K^2 \underbrace{\int_0^K E(k)\,dk}_{\text{always finite}}$$

$$\Rightarrow \lim_{\nu \to 0} \partial_t \mathcal{E}_K = \int_0^K T(k)\,dk \equiv -\Pi(K)$$

flux of energy due to non linear term.

$$T(K) = -\frac{\partial \Pi(k)}{\partial k}$$

$$\partial_t \mathcal{E}_K = -2\nu \Omega_K - \Pi_K + \mathcal{F}_K$$

if there is an external force

(i) consider stationary state

(ii) $K \to \infty$ for fixed ν

$\Rightarrow$ $\Pi_K \to 0$ because the non-linear term conserves energy.

$$\Rightarrow \underbrace{\mathcal{F}_\infty}_{\text{rate of energy injection}} = \underbrace{2\nu\Omega}_{\text{rate of energy dissipation}} = \varepsilon(\nu)$$

the equality is obvious.

• For a fixed K, with $K > K_{\text{injection}}$ (~~such that~~ such that $\mathcal{J}_K = \mathcal{J}_\infty$)

$$\partial_t \mathcal{E}_K = -2\nu\Omega_K - \Pi_K + \varepsilon(\nu)$$

Now consider stationarity and take limit ~~to~~ $\nu \to 0$

$$\Rightarrow \boxed{\lim_{\nu\to 0} \Pi_K = \varepsilon}$$

constant (By dissipative anomaly)

In the "turbulent" limit energy flux through K is constant and equal to ε.

Remember

$$\Pi_K = -\int_0^K T(k)\,dk = \int_K^\infty T(k)\,dk$$

$$\left[\text{because} \quad \int_0^\infty T(k)\,dk = 0 \right]$$

$$= \int_0^K \left[\int_{-\infty}^{+\infty} \delta(\vec{k}+\vec{p}+\vec{q})\, 2\pi k^2\, s(k,p,q)\, d\vec{p}\, d\vec{q} \right] dk$$

$$s(k,p,q) = P_{\alpha\beta\gamma}(k)\, \Im m \Big[\underbrace{\langle \hat{u}_\alpha(k)\, \hat{u}_\beta(p)\, u_\gamma(q) \rangle}_{\text{a third order quantity.}}$$

Count dimensions in the eqn:

$$\lim_{\nu\to 0} \Pi_K = \varepsilon$$

$$\int_0^K \underbrace{P_{\alpha\beta\gamma}(k)}_{k^1} \int \langle \hat{u}\,\hat{u}\,\hat{u} \rangle \, k^2 \, \underbrace{dp\,dq}_{k^6} \, \underbrace{dk}_{k^1} = \Big($$ is only a function of

because everything is integrated over

The third order structure function

$$S_3(\ell) = \langle [\delta u_{\parallel}(\ell)]^p \rangle$$

$$\sim \widetilde{FFT}\left[\langle \hat{u}\,\hat{u}\,\hat{u} \rangle\right]$$

$$\int \underbrace{\langle \hat{u}\,\hat{u}\,\hat{u} \rangle}_{u^3} \underbrace{dp\,dq\,dk}_{k^9}$$

Dimensionally

$$k\, S_3(\ell) \sim \varepsilon \qquad k \sim K \sim \frac{1}{\ell}$$

$$\Rightarrow \quad \cancel{S_3(\ell)} \sim \cancel{\ell}$$

$$\Rightarrow \quad \boxed{S_3(\ell) \sim \varepsilon \ell.}$$

The constancy of flux in fourier space implies that the third order structure function is propor to ℓ. This can be made into an exact (the only exact relation) in turbulence

$$\boxed{S_3(\ell) = -\frac{4}{5} \varepsilon \ell}$$

Kolmogorov's 4/5 th law.

negative.

$\Rightarrow$ energy goes from large to small scales small to large k

~~Kolm.~~

Phenomenology

- Dissipation length $\quad \eta \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$

- Taylor microscale

- characteristic time scales, lifetime of eddies.

- Univasality, Kolmogorov constant,
  Landau's comment.

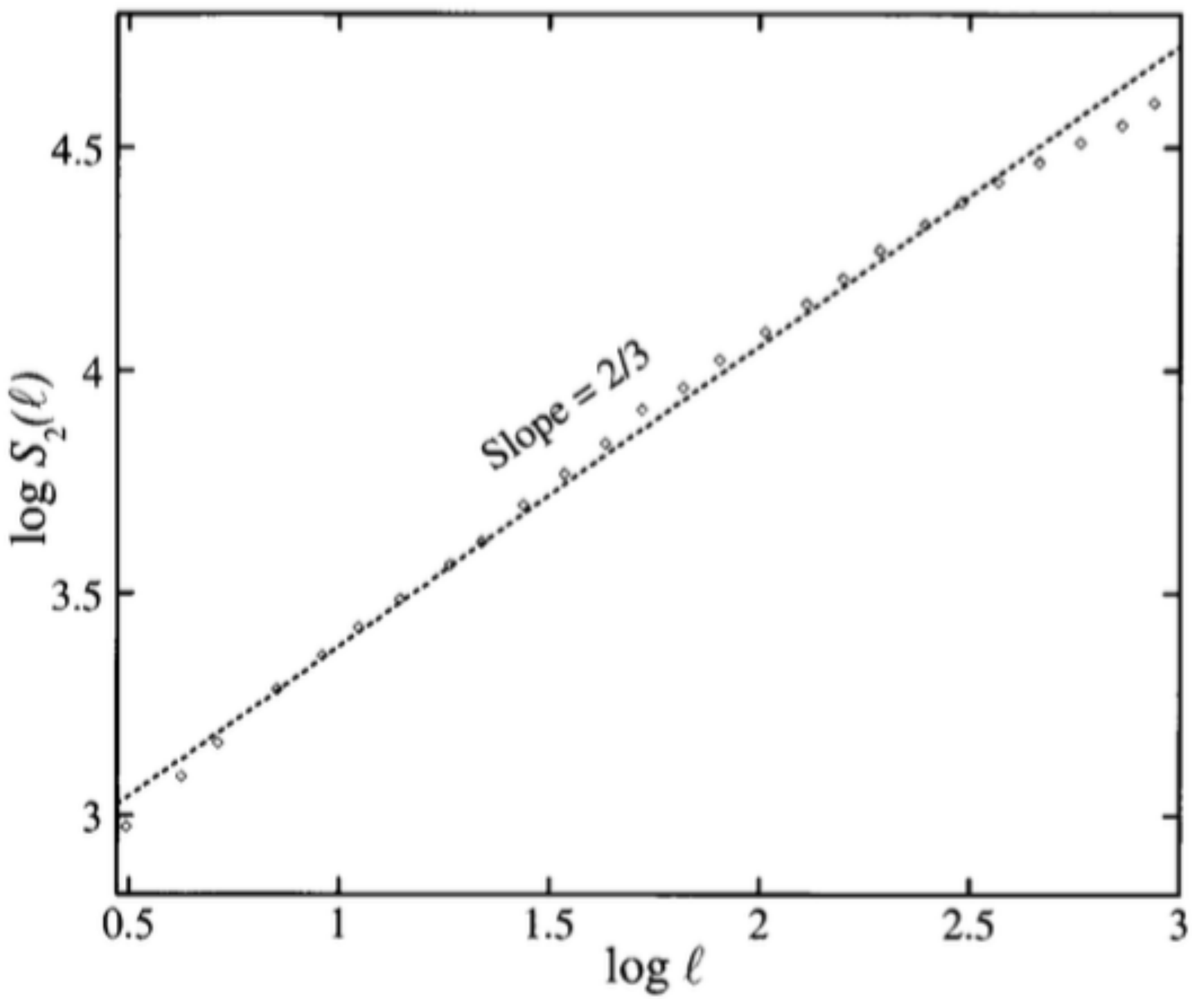
- Kolmogorov is not Gaussian. (Third order mom
  is not zero) But ~~all~~ ~~ht~~ the scaling of
  higher moments are determined by ~~ϕ~~ only
  moment.

Intermittency :

- When is a signal intermittent ?

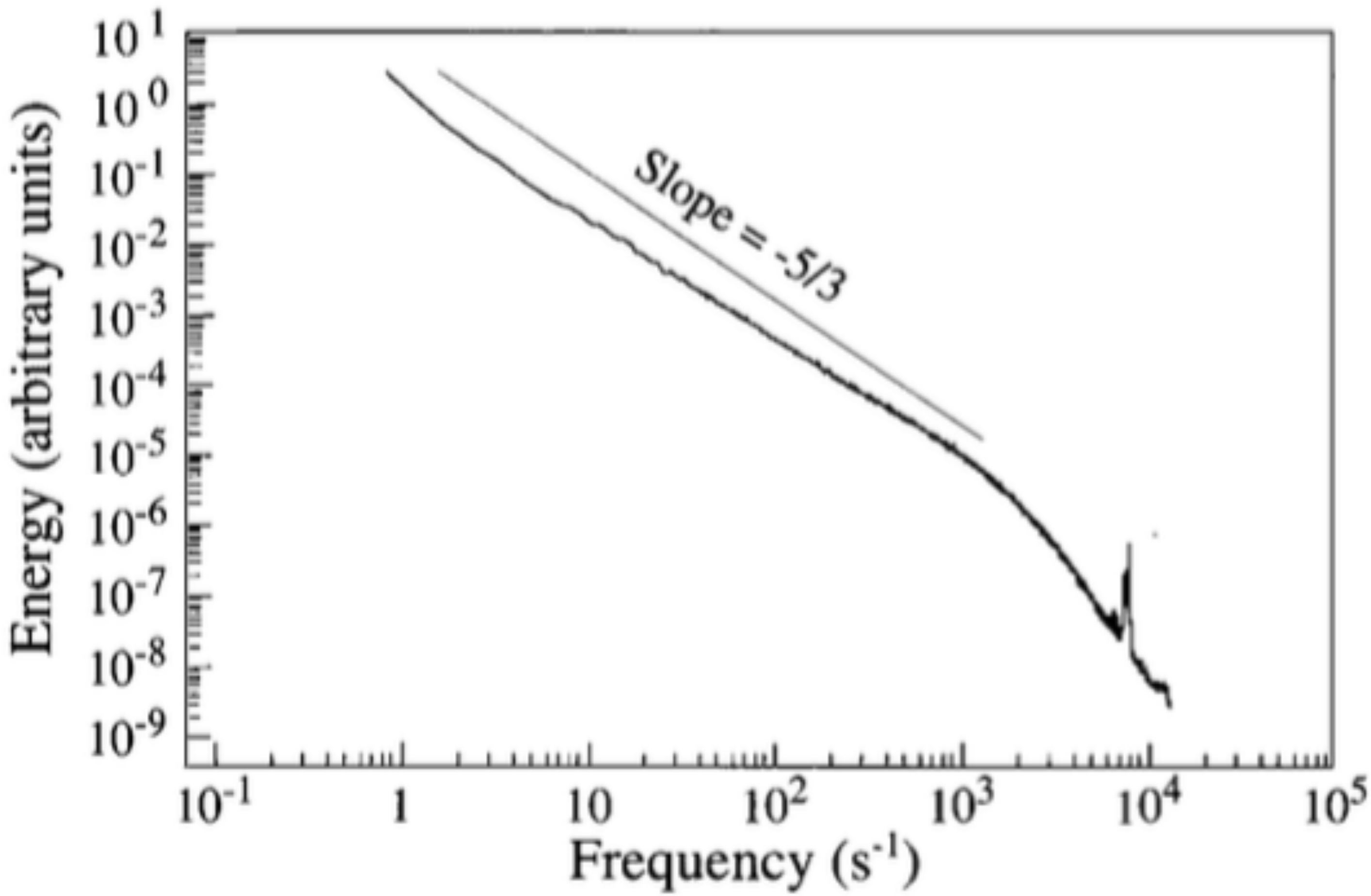
Closure and EDQNM

Experimental evidence



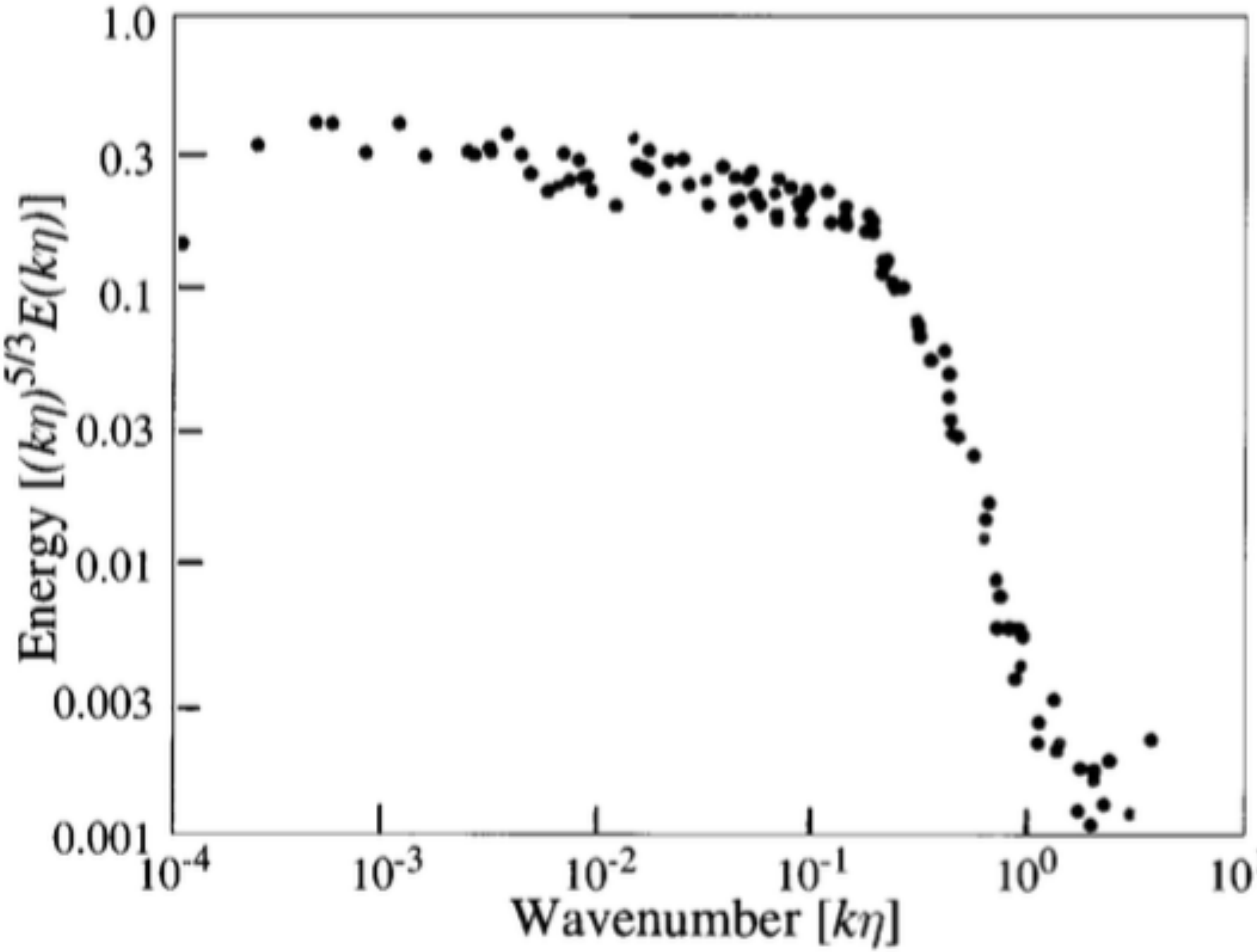
Data from wind tunnel ONERA

Experimental evidence



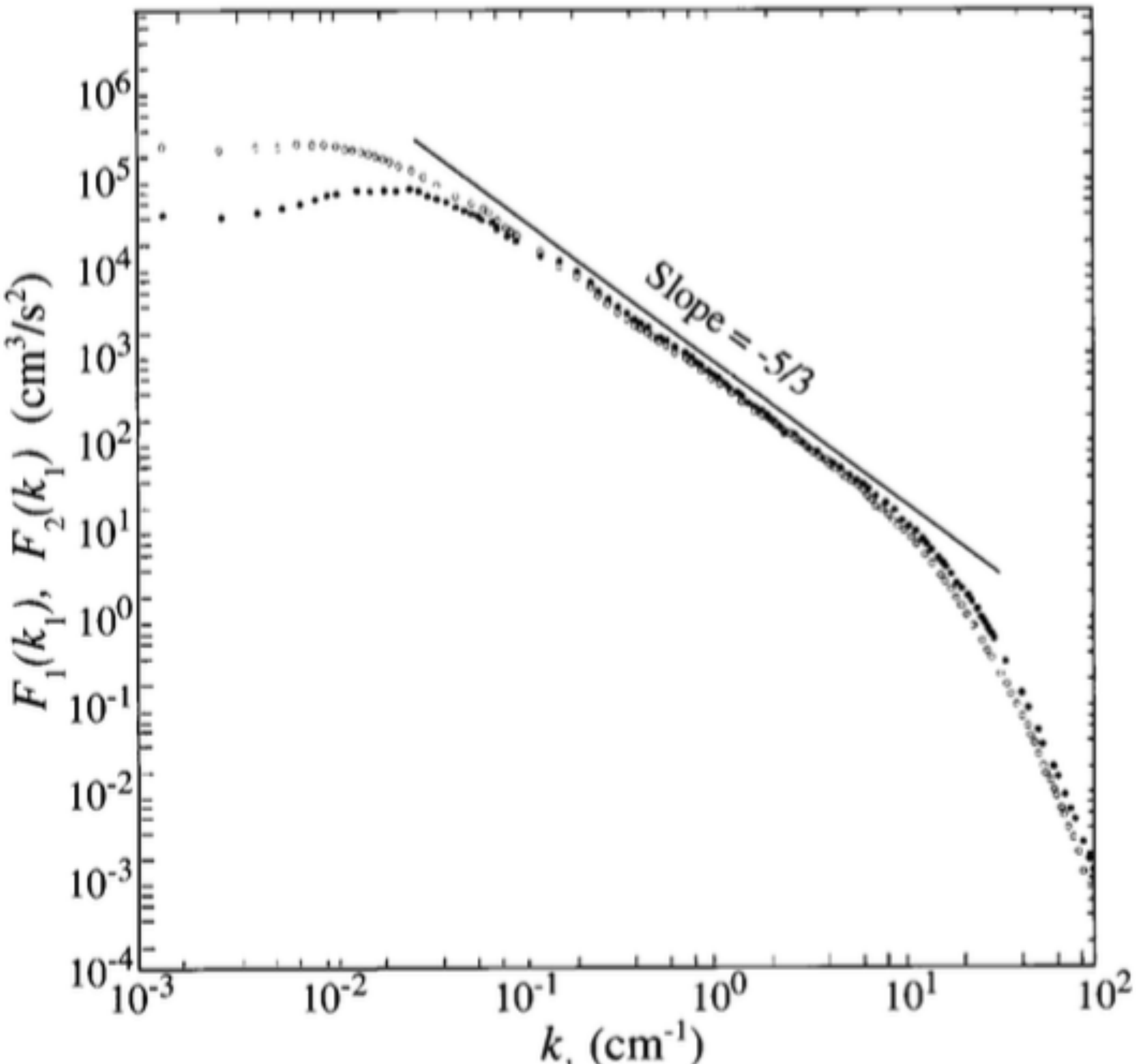
Data from wind tunnel ONERA, energy spectrum

Experimental evidence



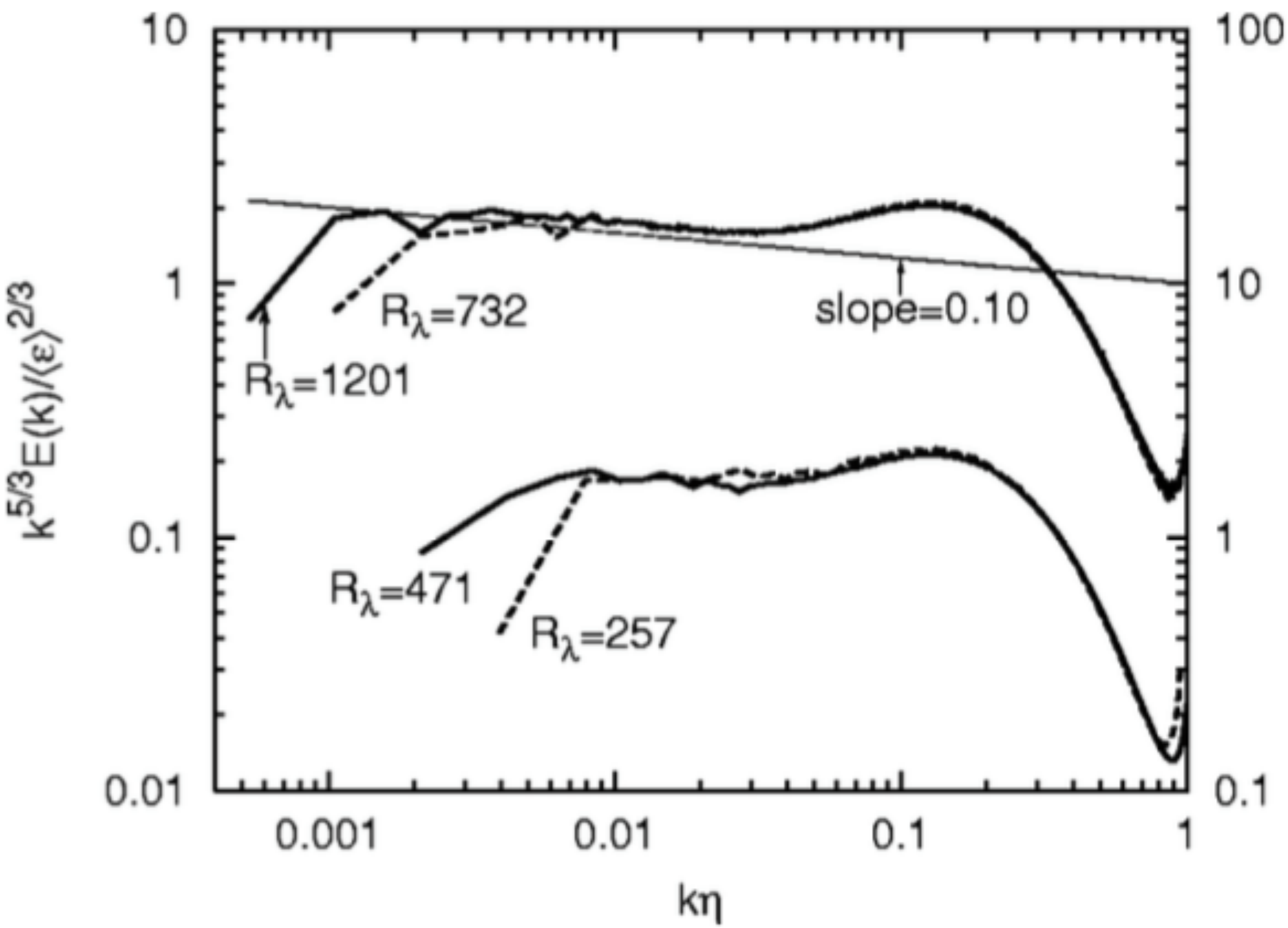
Compensated spectra from tidal channel

Experimental evidence



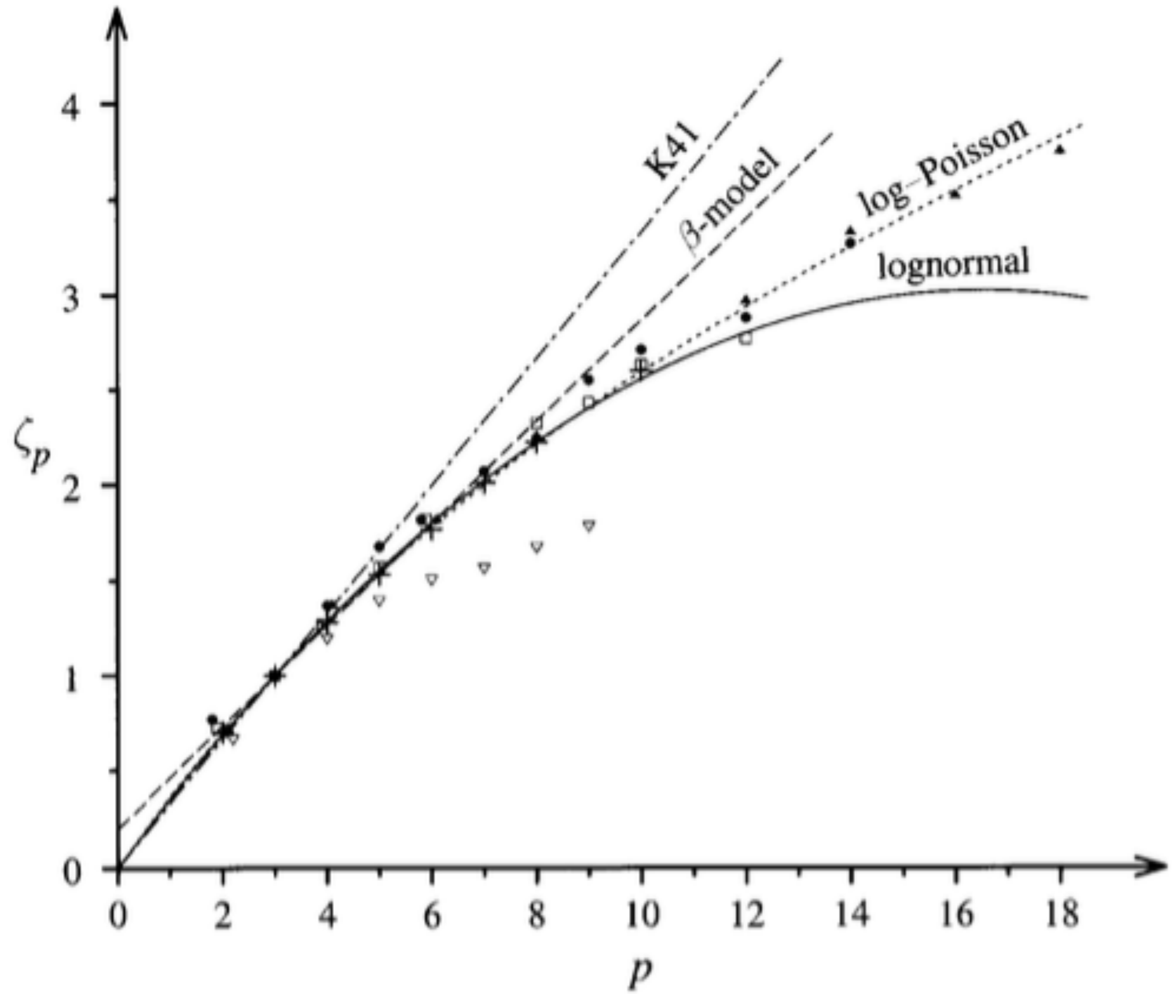
Velocity fluctuations (in time domain) in a jet (Champagne 1978)

Numerical evidence



Biggest numerical simulations so far (4096 cubed), Kaneda et al 2003

Intermittency



Biggest numerical simulations so far (4096 cubed), Kaneda et al 2003

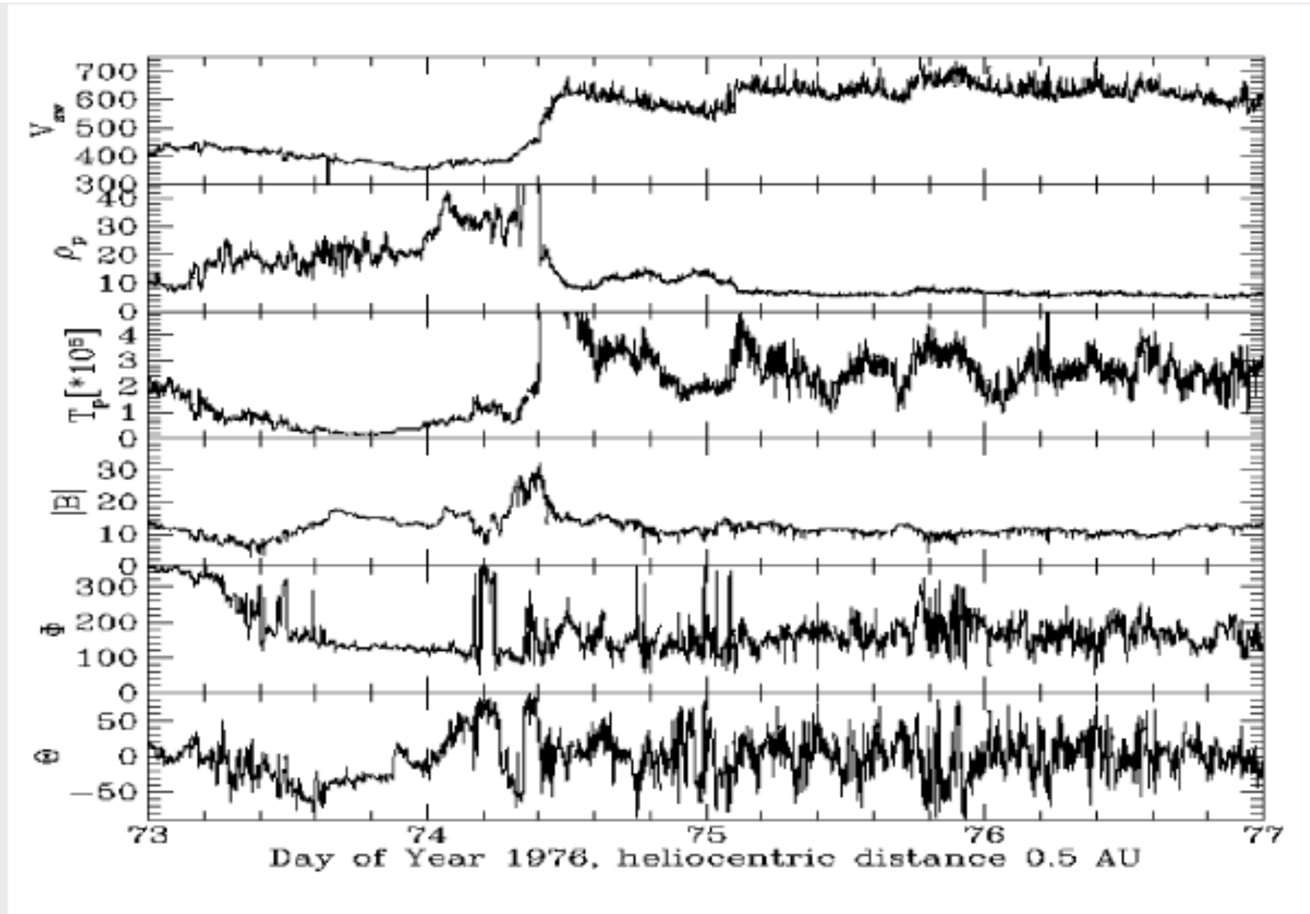
Turbulence in solar wind

- Compressible and with magnetic field. This implies that Kolmogorov's theory, as we have described, does not apply. It must be extended.
- We need new relations to replace the Karman-Howarth-Monin relation.

$$S_3(\ell) = -\frac{4}{5}\varepsilon\ell$$

- This is not trivial. In incompressible MHD this was first worked out by Chandrasekhar. A similar relation was worked out by Politano and Pouquet which is possibly wrong. We shall not get into this at the moment.
- In compressible turbulence (not MHD) an equivalent relation has been worked out by Banerjee and Galtier.

Solar wind turbulence (experiments)



- There is fast and slow wind. The wind is turbulent, and also "intermittent". (wind during solar minima, Bruno and Carbone, living reviews in solar physics)

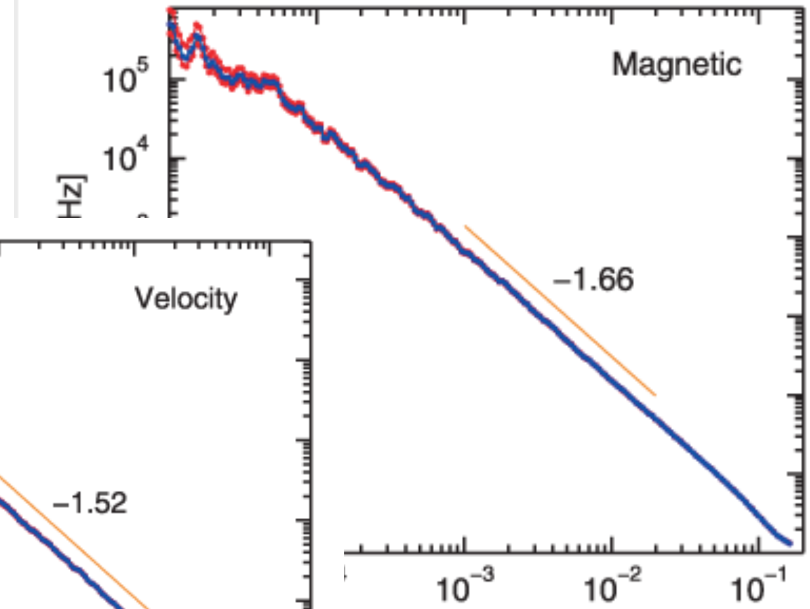
spectra

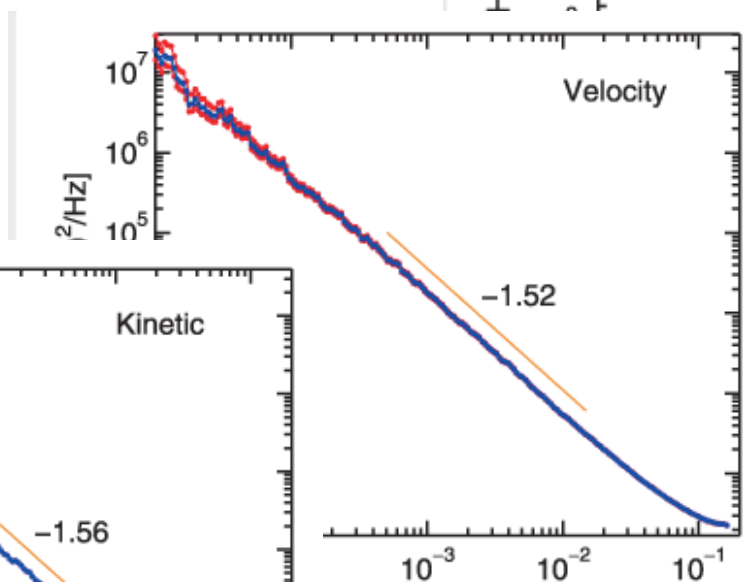
$$E(k) \sim k^{-3/2}$$

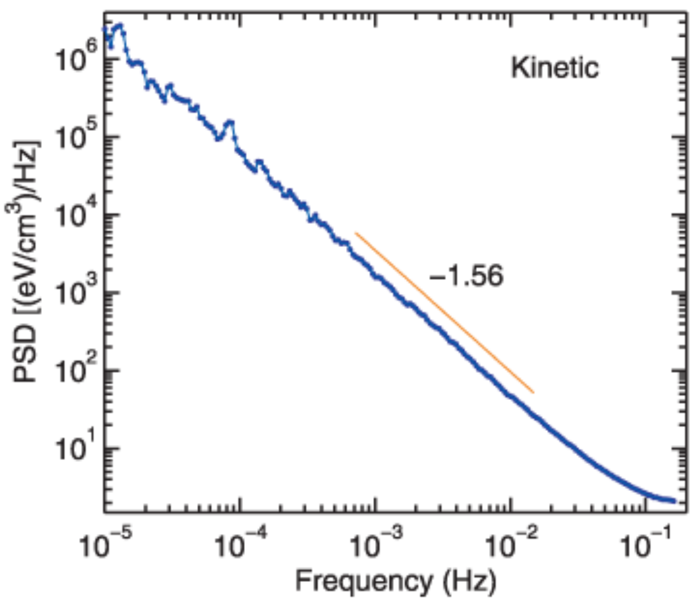
Irishnikov-Kraichnan

$$E(k) \sim k^{-5/3}$$

Kolmogorov







$$E_\perp(k) \sim k_\perp^{-2}$$

- Modes perpendicular to mean magnetic field.
- Weak turbulence theory, Goldrich-Sridhar scaling.
- Galtier et al, Journal of Plasma Physics, 2000

- We have no theory of this.

EDQNM

$$\left[\partial_t + \nu(k_1^2 + k_2^2)\right]\langle \hat u(k_1)\, \hat u(k_2)\rangle = \langle \hat u \hat u \hat u \rangle$$

$$\left[\partial_t + \nu(k_1^2 + k_2^2 + k_3^2)\right]\langle \hat u(k_1)\, \hat u(k_2)\, \hat u(k_3)\rangle$$

$$= \langle \hat u \hat u \hat u \hat u\rangle$$

$$= \sum (\quad)\langle \hat u \hat u\rangle\langle \hat u \hat u\rangle$$

combinatorial factor

Assuming the PDF to be Gaussian. Quasi-Normal approximation

$$\not\Rightarrow \quad \cancel{\hat u(k_1)\, \hat u(k_2)\, \hat u(k_3)}$$

Notation

$$\left[\partial_t + \nu(k_1^2 + k_2^2)\right]\hat U_2(k_1, k_2) = \hat U_3(k_1, k_2; k_3) \quad \nexists$$

$$\left[\partial_t + \nu(k_1^2 + k_2^2 + k_3^2)\right]\hat U_3(k_1, k_2, k_3)$$

$$= \hat U_4(k_1, k_2, k_3, k_4)$$

$$= \sum \hat U_2(k_1, k_2)\, U_2(k_3, k_4) + (\text{permutation})$$

$$\Rightarrow \quad \hat U_3(k_1, k_2, k_3) = \int \left[\hat U_2(k_1, k_2)\, \hat U_2(k_3, k_4) + \text{permutations}\right] e^{-\nu(k_1^2 + k_2^2 + k_3^2)\tau}\, d\tau$$

$$\Rightarrow \quad \left[\partial_t + \nu(k_1^2+k_2^2)\right] \hat{U}_2(k_1, k_2)$$

$$= \int d\tau \; e^{-\nu\left(k_1^2+k_2^2+k_3^2\right)\tau} \; \delta(k_1+k_2-k_3)$$

$$\left[\hat{U}(k_1, k_2)\,\hat{U}(k_3, k_4) + \text{permutation}\right]$$

A closed equation at ~~second~~ third order. This is an example of closure.

Integrating over angular variables and simplifying one obtains

$$\left[\partial_t + 2\nu k^2\right] E(k, t) = \int_0^t d\tau \int e^{-\nu(k_1^2+k_2^2+k_3^2)(t-\tau)} \, S(k_1, k_2, k_3) \, dk_1 \, dk_2$$

$$S(k, k_2, k_3) = \frac{k^3}{k_2 k_3} \, a(k, k_2, k_3) \, E(k_1) \, E(k_2)$$

$$- \text{ other quadratic terms in } E$$

This equation can be solved numerically to obtain the spectrum.

It turns out that the numerical solutions have negative energy. So the closure gives unphysical answer.

Solution :

Add an "eddy damping" term

$$\left[\partial_t + \nu(k_1^2 + k_2^2 + k_3^2) + \mu_{k_1 k_2 k_3}\right] \hat{U}(k_1, k_2, k_3)$$

$$= \sum_{\text{permutations}} \hat{U}(k_1, k_2)\, \hat{U}(k_2, k_3)$$

$$\mu_{k_1 k_2 k_3} = \mu_{k_1} + \mu_{k_2} + \mu_{k_3}$$

$$\mu_q \simeq \left[q^3 E(q)\right]^{1/2}$$ for isotropic case.

But the positiveness of energy spectra is still not guaranteed.

Solution : Markovization :

$$(\partial_t + 2\nu k^2)\, E(k,t) = \int \theta_{kpq} \sum \hat{U}\hat{U} \; dp\,dq$$

$$\theta_{kpq} = \int_0^t e^{-\mu_{kpq} + \nu(k^2 + p^2 + q^2)(t-\tau)}\, d\tau$$

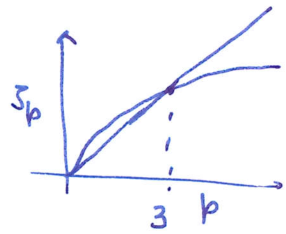
not a function of time any more.

The problem of turbulence

- summary of known results : $S_3 = -\frac{4}{5}\varepsilon l$

  ~~of~~ $S_p(l) \sim l^{\zeta_p}$

K41



- How do we make a theory of intermittency? Such a theory ~~sh~~ should start ~~for~~ from the Navier-Stokes equation and give us the exponents ζ_p. This is the problem of turbulence.

- what pieces do we know?

  (a) At small scales $S_p(l) \sim l^p$. i.e the structure functions ~~any~~ are Taylor expandable.

  $\Rightarrow$ At large scales $\sim e^{-(\delta k)^2}$

  $E(k) \sim e^{-(\delta k)^2}$ at large k

  where δ is the distance to the nearest singularity in the complex plane.

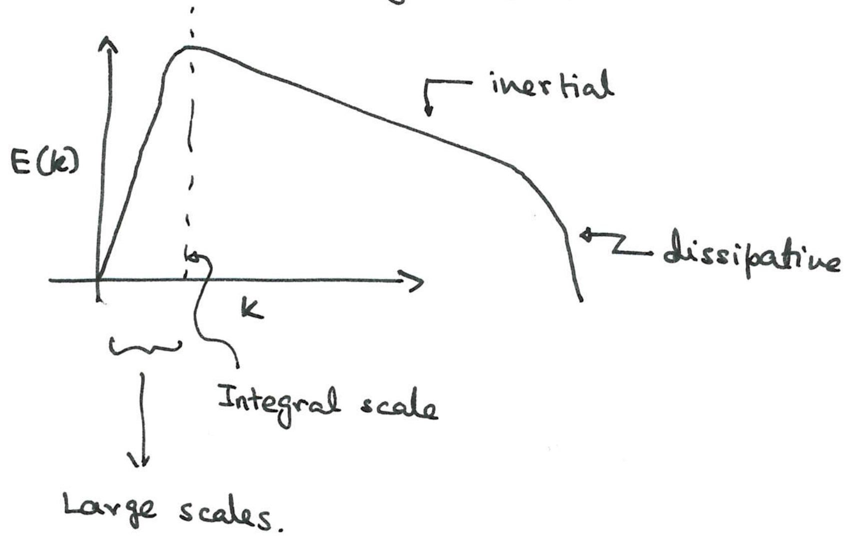
  (b) In a model of passive scalar, — a model that is linear but stochastic — one can carry out the program of calculation of the anomalous exponents.

- How is this related to the famous problem of singularities of the Navier-Stokes eqn?

Not in a very obvious way. The intermittency does not imply singular structures but singular behaviour on "average".

Lecture X

Large scale turbulent dynamos :



Question :
How can we extract large scale patterns from turbulence?

Answer By averaging the equations of MHD to write an effective equation for large scale behaviour. Such equations may be ~~more~~ nastier than the MHD equations themselves and may have to be solved numerically.

- ~~Ave~~
- Reynold's Averaging :

$$\overline{U_1 + U_2} = \overline{U}_1 + \overline{U}_2 \quad ; \quad \overline{\overline{U}} = \overline{U}$$

$$\overline{\partial U/\partial t} = \partial \overline{U}/\partial t \quad , \quad \overline{\nabla U} = \nabla \overline{U}$$

$$\overline{u} = 0 . \quad \overline{b} = 0$$

~~They~~ ' The Reynold's rules are satisfied by ~~the~~ averaging over one or more coordinate direction but not Fourier filtering. To see how it works let us apply this to the induction equation

$$\partial_t B = \nabla \times (U \times B) + \nabla \times (-\eta J)$$

$$B = \bar{B} + b$$

$$U = \bar{U} + u \quad \Rightarrow \quad J = \bar{J} + j$$

with $\bar{J} = \nabla \times \bar{B}$

and $j = \nabla \times b$

Average the whole eqn:

$$\partial_t \bar{B} = \nabla \times \overline{U \times B} + \nabla \times (-\eta \bar{J})$$

$$U \times B = (\bar{U} + u) \times (\bar{B} + b)$$

$$= \bar{U} \times \bar{B} + u \times \bar{B} + \bar{U} \times b + u \times b$$

$$\Rightarrow \quad \overline{U \times B} = \bar{U} \times \bar{B} + \overline{u \times b}$$

$$\Rightarrow \quad \underbrace{\partial_t \bar{B} = \nabla \times \left(\bar{U} \times \bar{B} - \eta \bar{J} \right)}_{\text{closed}} + \underbrace{\nabla \times \overline{u \times b}}_{\text{not closed.}}$$

$$\cancel{\mathcal{E}} = \cancel{\nabla\times} \qquad \bar{\mathcal{E}} \equiv \underbrace{\overline{u \times b}}_{\text{A term we need to model.}}$$

we demand closure; i.e.

$$\mathcal{E}_i = \alpha_{ij} \bar{B}_j + \beta_{ijk} \partial_j \bar{B}_k + \left(\text{higher order terms} \right)$$

when $\bar{U} = 0$.

Otherwise there would be other coefficients multiplying $\bar{U}$. α_{ij}, β_{ijk} are "turbulent transport coefficients."

- How to calculate the transport coefficients?

$$\partial_t b = \nabla \times \left(\underbrace{\bar{U} \times b}_{\substack{\text{assume } \bar{U}=0 \\ \text{(simple case)}}} + u \times \bar{B} + u \times b - \bar{\mathcal{E}} - \eta j \right)$$

Now calculate:

$$\partial_t \overline{\mathcal{E}} = \partial_t \overline{u \times b}$$

$$= \overline{\partial_t u \times b} + \overline{u \times \partial_t b}$$

First work in the kinematic approximation:

$$\partial_t U = -U \cdot \nabla U - \nabla p + F_{visc} + f + \underbrace{(J \times B)}_{\text{ignored.}}$$

obtain:
$$\partial_t u = \partial_t \overline{U} - \partial_t U$$

After substitution and simplifications:

$$\partial_t \overline{\mathcal{E}} = \overline{u \times \nabla \times (u \times \overline{B})} + \left(\begin{array}{c}\text{triple} \\ \text{correlations}\end{array}\right)$$

Note that terms like

$$\overline{u \times \nabla \times \overline{\mathcal{E}}} = \overline{u} \times \overline{\nabla \times \overline{\mathcal{E}}} = 0 \quad (\text{as } \overline{u} = 0)$$

on simplification:

$$\partial_t \overline{\mathcal{E}} = \tilde{\alpha}\, \overline{B} - \eta_t \overline{J} - \underbrace{\frac{\overline{\mathcal{E}}}{\tau}}_{\text{effect of all the triple correlations}}$$

The last term is a closure hypothesis.

Next we assume $\partial_t \overline{\mathcal{E}} = 0$ is in the stationary state. This assumption is also tricky.

where we obtain:

$$\left.\begin{aligned} \tilde{\alpha} &= -\frac{1}{3}\,\overline{\omega\cdot u} \\ \tilde{\eta}_t &= \frac{1}{3}\,\overline{u^2} \end{aligned}\right\}$$ turbulent transport coefficients.

<u>Exercise</u>: show this.

- <u>comments</u>

- For a non-zero $\tilde{\alpha}$

$$\partial_t \bar{B} = \nabla \times \left(\tilde{\alpha}\, \bar{B} - \eta_t \bar{J}\right)$$

we can obtain exponential growth of $\bar{B}$.

This is the dynamo effect.

- But $\tilde{\alpha}$ is non-zero only when kinetic helicity $\mathcal{H} = \overline{\omega\cdot u}$ is non-zero.

We need helical flows to generate large scale magnetic fields.

There is another way to derive the same result:

Start with $$\partial_t b = \nabla\times\left(u\times B - u\times b - \overline{\mathcal{E}} - \eta j\right)$$

$$\simeq \nabla\times(u\times B)$$

ignore the others because:

(a) $\overline{u\times b}$ is one order higher in fluctuations

(b) ~~so~~ Lator $\overline{u\times\overline{\mathcal{E}}} = 0$

(c) η is very small.

Then $$b(t) = \int_0^t \nabla\times(u\times B)\, dz$$

$$\Rightarrow \quad \overline{\mathcal{E}} = \overline{u\times b}$$

$$= \int_0^t \overline{u\times\nabla\times(u\times B)}\, dz$$

$$= \alpha B - \eta_t J$$

This is called the first order smoothing approximation and the earlier one is called the minimal tau approximation. They are not exactly the same. But quite similar.

Mean field theory in general:

$$\partial_t(\rho U) + \mathrm{div}\left(\rho\, U_i U_j - \delta_{ij}\, p + \sigma_{ij} - \delta_{ij}\frac{B^2}{\mu_0} + \frac{B_i B_j}{\mu_0}\right) = 0$$

• How does one do Reynold's averaging with density:

$$\overline{\rho U} = \bar{\rho}\,\bar{U} \; ?$$

For the moment consider incompressible flows, $\Rightarrow \rho = \text{const}$

Then

$$\partial_t \overline{\rho U} + \mathrm{div}\left(\rho\,\overline{U_i U_j} - \delta_{ij}\,\bar{p} + \bar{\sigma}_{ij} - \delta_{ij}\frac{\overline{B^2}}{\mu} + \frac{\overline{B_i B_j}}{\mu_0}\right) = 0$$

$\overline{U_i U_j} \equiv$ Reynold's stress

$\frac{1}{\mu_0}\overline{B_i B_j} \simeq$ Maxwell's stress.

In the sense of closure:

$$\overline{U_i U_j} = (\,)\,\bar{U} + (\,)\,\nabla U + \left(\text{higher order term}\right)$$

Can be a problem because can violate Galilean invariance.

turbulent diffusivity.

$$= \eta^t_{ijkl}\,\partial_k \overline{U_l}$$

- The heart of the problem is to calculate the turbulent-transport coefficient, α_{ij}, η^t_{ij}, ν^t_{ijkl} and many others. There is no systematic way to calculate them at high Reynold's number. They can either be calculated analytically by uncontrolled closure. Or calculated numerically.

Exercise 10 — 25 marks

Please return on 22nd March

1. Consider a case where an incompressible fluid of variable density ~~is at an~~ under gravity has a stationary solution to the equation of motion with $\rho_0 = \rho(z)$ and $\vec{v}_0 = 0$. Let the three components of velocity be (u, v, w). Linearize the equation of motion and show that the linearized equation of motion is

$$\rho_0 \partial_t u = -\frac{\partial}{\partial x}\delta p + \mu \nabla^2 u + \boxed{\left(\cancel{\frac{\partial w}{\partial x}} + \cancel{\frac{\partial u}{\partial z}}\right)}$$

$$\rho_0 \partial_t v = -\frac{\partial}{\partial y}\delta p + \mu \nabla^2 v$$

$$\rho_0 \partial_t w = -\frac{\partial}{\partial z}\delta p + \mu \nabla^2 w - g\,\delta\rho$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\partial_t \delta\rho = -w\frac{d\rho_0}{dz}$$

Here we have assumed that the perturbations of velocity are (u, v, w) and the perturbations of density and pressure are $\delta\rho$ and δp. The dynamical viscosity μ is constant. g is the gravitational acceleration.

Seek solutions of the form

$$\exp i(k_x x + k_y y + nt)$$

Then show that

$$\boxed{\cancel{D\left\{\left[\rho - \frac{\mu}{n}(D^2-k^2)\right]D\omega - \frac{1}{n}(D\mu)\,D\right.}}$$

$$D\left[\rho_0 - \frac{\mu}{n}(D^2-k^2)\right]D\hat{\omega}$$

$$= k^2\left\{-\frac{g}{n^2}(D\rho)\hat{\omega} + \left[\rho - \frac{\mu}{n}(D^2-k^2)\right]\hat{\omega}\right\} \qquad -(1)$$

where $D = \frac{d}{dz}$, $k^2 = k_x^2 + k_y^2$

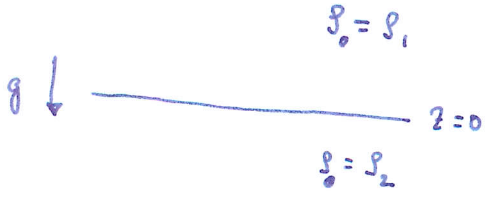
and $\omega = \hat{\omega}(z)\exp i(k_x x + k_y y + nt)$

Now consider the inviscid case : $\mu = 0$

~~wrote down the~~ and show that the above relation simplifies to

$$D(\rho D\hat{\omega}) - \rho k^2\hat{\omega} = -\frac{k^2}{n^2}g(D\rho)\hat{\omega} \qquad -(2)$$

Now further simplify the problem to the case of two fluids of density ρ_1 and ρ_2 separated by a boundary at $z=0$. Ignore surface tension such that the above equations apply.



The way to solve this problem is to apply Eq. (2) ~~$\ast$~~ separately to $z>0$ and $z<0$. And then match the solutions at $z=0$.

Show that for $z > 0$ (or $z < 0$) Eq. (2) reduces to:

$$(D^2 - k^2)\,\hat{\omega} = 0$$

solve this with the boundary condition $\hat{\omega} \to 0$ as $z \to +\infty$

similarly for $z < 0$, solve the same eqn. with boundary condition $\hat{\omega} \to 0$ as $z \to -\infty$. Then assume $\hat{\omega}$ should be continuous at $z = 0$.

Then $$w = A e^{kz} \quad (z < 0)$$
$$= A e^{-kz} \quad (z > 0)$$

2. The equation obeyed by a passive scalar in a flow is

$$\partial_t \theta + (u.\nabla)\theta = \kappa \nabla^2 \theta$$

Assuming $$\theta(x) = \int \hat{\theta}(k)\, e^{ikx}\, dk$$

$$u(x) = \int \hat{u}(k)\, e^{ikx}\, dk$$

write down the equation satisfied by $\hat{\theta}(k)$.

3. Consider the passive scalar equation

$$\partial_t \theta + (\mathbf{U}.\nabla)\theta = \kappa \nabla^2 \theta$$

- assume $\mathbf{U}$ is incompressible
- Next do mean-field decomposition

$$U = \bar{U} + u$$

$$\theta = \bar{\theta} + \phi$$

Demand that the equation at large scale will be given by the closure:

$$\overline{u_j \phi} = K_{ij}\, \partial_j \bar{\theta}$$

Then show, using FOSA as described in class that

$$K_{ij} = \int \overline{u_i(t)\, u_j(t)}\, dt$$

and

$$\partial_t \bar{\Theta} = \mathrm{div}\left(\kappa \nabla \bar{\Theta} + K_{ij}\, \partial_j \bar{\Theta}\right)$$

comment on why there is no alpha effect here?

(• Assume $\bar{U} = 0$)

4. Consider the dynamo problem in a case where its axisymmetric. In spherical coordinates

$$B = B_\phi(r,\theta)\,\hat{e}_\phi + B_p$$

Where B_ϕ is the toroidal component and B_p is the poloidal component. Write

$$B_p = \nabla \times \left[A_\phi(r,\theta)\, e_\phi\right]$$

and write the velocity field as

$$v = \Omega(r,\theta)\, r \sin\theta\, \hat{e}_\phi$$

Show that the ~~gover~~ ~~inducti~~ ~~d~~ mean field dynamo eqn.

$$\partial_t B = \nabla \times \left(\alpha B - \eta_T J\right)$$

with constant α and η_T reduces to:

$$\partial_t B_\phi = r\sin\theta \left(B_p \cdot \nabla\right)\Omega + \hat{e}_\phi \cdot \left[\nabla \times \left(\alpha B_p\right)\right] + \eta_T\left(\nabla^2 - \frac{1}{r^2 \sin^2\theta}\right) B_\phi$$

~~&~~ $$\partial_t A_\phi = \alpha B_\phi + \eta\left(\nabla^2 - \frac{1}{r^2 \sin^2\theta}\right) A_\phi$$

5. The solar dynamo has a period of 22 years and its half wavelength corresponds to about 40° in lattitude. Assuming the dynamo to be an alpha-shear dynamo make a rough estimate of the quantity (αG) [where G is the shear] and turbulent diffusion coefficient η_T. Assume the dynamo to be maginally stable.