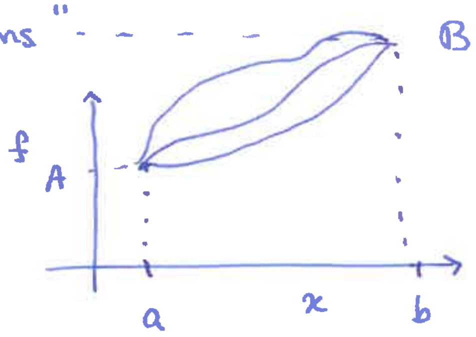
Functionals :

"function of ~~a~~ functions"



1. Example :

$$\mathcal{F}[f] = \int_a^b f \, dx$$

$$\mathcal{F}[f] = \int f(x)\, \delta(x-y)\, dx$$

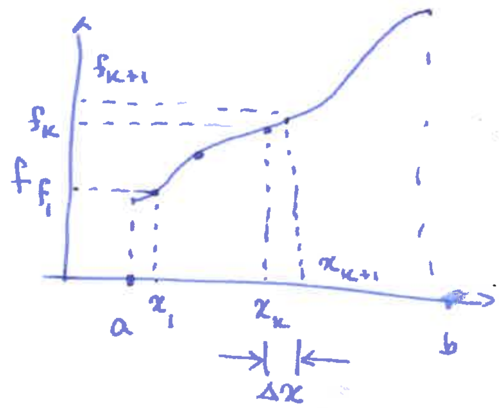
$$\mathcal{F}[f] = \int f^2 \, dx$$

$$\vdots$$

2. Functionals ~~and~~ the are continuum limit of functions of many variables.

$$\mathcal{F}[f] = \int_a^b F(x, f)\, dx$$

$$= \lim_{\substack{N \to \infty \\ \Delta x \to 0}} \sum_{k=1}^{N} \Delta x \; F(x_k, f_k)$$



3. Derivative of functionals.

To define the derivative of a function:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

distance

what is the concept of "distance" between function?

To define a distance we need to define a "norm". One choice of norm in function space could be:

$$\| f \| \equiv \max_{x \in [a,b]} | f(x) |$$

Then all functions f are in an ε neighbourhood of a function f_* iff

$$\| f - f_* \| \leq \varepsilon$$

$$\Rightarrow \quad \max_{x \in [a,b]} | f - f_* | \leq \varepsilon$$

A good norm for all functions that are continuous in the domain $x \in [a, b]$

But if we want a different class of functions; e.g., all function within the domain $x \in [a,b]$ that are continious and once differentiable then we need a different norm, e.g,

$$\|f\|_1 \equiv \max_{x \in [a,b]} |f(x)| + \max_{x \in [a,b]} |f'(x)|$$

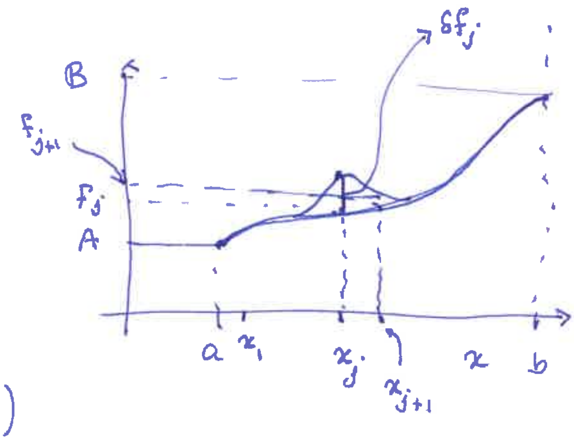
Two functions are neighbours only if both their maximum value and the maximum value of their derivatives are close together.

Now we are ready to define continuity :

Derivatives :

Start by considering $\mathcal{J}$ as a function of N variables $(f_1, \ldots, f_j, \ldots f_N)$

Then the functional $\mathcal{J}$ becomes



$$\mathcal{J}_N(f_1, \ldots f_j, \ldots f_N) = \sum_{j=1}^{N} \Delta x \; F(x_j, f_j, f_j')$$

where $f_j' = \left.\dfrac{df}{dx}\right|_{x=x_j} = \dfrac{f_{j+1} - f_j}{\Delta x}$

Now calculate

$$\frac{\partial \mathcal{J}_N}{\partial f_\ell} = \sum \Delta x \left[\frac{\partial F}{\partial f_j}\frac{\partial f_j}{\partial f_\ell} + \frac{\partial F}{\partial f_j'}\frac{\partial f_j'}{\partial f_\ell} \right]$$

Clearly the variation ~~at~~ at f_j is independent ~~of~~ of f_ℓ

$$\Rightarrow \quad \frac{\partial f_j}{\partial f_\ell} = \delta_{j\ell}$$

and

$$\frac{\partial f_j'}{\partial f_\ell} = \frac{1}{\Delta x}\left(\frac{\partial f_{j+1}}{\partial f_\ell} - \frac{\partial f_j}{\partial f_\ell}\right)$$

$$= \frac{1}{\Delta x}\left(\delta_{j+1,\ell} - \delta_{j,\ell}\right)$$

Substituting back we obtain:

$$\frac{\partial \mathcal{F}_N}{\cancel{\Delta x}\,\partial f_\ell} = \sum_N \frac{\partial F}{\partial f_j}\delta_{j\ell}\Delta x + \sum_N \cancel{\Delta x}\,\frac{\partial F}{\partial f'_j}\left(\delta_{j+1,\ell} - \delta_{j,\ell}\right)$$

$$= \frac{\partial F}{\partial f_\ell}\Delta x + \left(\frac{\partial F}{\partial f'_{\ell-1}} - \frac{\partial F}{\partial f'_\ell}\right)$$

$$= \frac{\partial F}{\partial f_\ell}\Delta x - \frac{d}{dx}\left(\frac{\partial F}{\partial f'_\ell}\right)(\Delta x)$$

where the second term on the RHS follows from Taylor expansion.

$$\Rightarrow \quad \frac{1}{\Delta x}\frac{\partial \mathcal{F}_N}{\partial f_\ell} = \frac{\partial F}{\partial f_\ell} - \frac{d}{dx}\left(\frac{\partial F}{\partial f'_\ell}\right)$$

Taking the continuum limit we obtain:

$$\cancel{\frac{\delta \mathcal{F}}{\delta f}} = \cancel{\frac{\partial F}{\partial f}}$$

$$\boxed{\frac{\delta \mathcal{F}}{\delta f(y)} = \frac{\partial F}{\partial f(y)} - \frac{d}{dx}\left(\frac{\partial F}{\partial f'(y)}\right)}$$

which is the well-known Euler-Lagrange eqn.

- Remark: the quantity $\Delta x \delta f_\ell$ is the element of an area.

Alternative derivation :

It is possible to use a more formal method that hides a lot of details under the carpet.

For example consider :

$$\mathcal{J}[f] = \int_a^b F(x, f, f')\, dx$$

& clearly the functional derivative must follow the following rule:

$$\frac{\delta f(x)}{\delta f(y)} = \delta(x-y)$$

This is a continuum version of $\frac{\partial f_j}{\partial f_l} = \delta_{jl}$ that we used ~~before~~. before.

Hence:

$$\frac{\delta \mathcal{J}}{\delta f(y)} = \int \frac{\partial F}{\partial f(x)} \frac{\delta f(x)}{\delta f(y)}\, dx + \int \cancel{\partial F} \frac{\partial \mathcal{J}}{\partial f'(x)} \frac{\delta f'(x)}{\delta f(y)}\, dx$$

$$\frac{\delta f'(x)}{\delta f(y)} = \frac{\delta}{\delta f(y)} \frac{df}{dx} = \frac{d}{dx} \frac{\delta f(x)}{\delta f(y)} = \frac{d}{dx} \delta(x-y)$$

this interchange of limits is allowed as x and f are varied independly

$$\Rightarrow \left(\begin{array}{c}\text{the second} \\ \text{term}\end{array}\right) = \int \frac{\partial \mathcal{J}}{\partial f'(x)} \frac{d}{dx} \delta(x-y)\, dx$$

$$= -\int \frac{d}{dx}\left(\frac{\partial \mathcal{J}}{\partial f'(x)}\right) \delta(x-y)\, dx$$

where the last step follows from integration by parts and throwing away the boundary terms. This we can do because we consider variations where the boundary is fixed.

$$\Rightarrow \quad \frac{\delta \mathcal{F}}{\delta f} = \frac{\partial F}{\partial f} - \frac{d}{dx}\left(\frac{\partial F}{\partial f'}\right)$$

The same Euler-Lagrange relation.