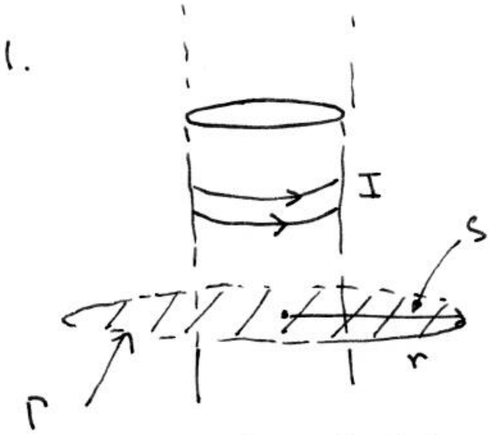
Solutions to problem set IV

1.



At any time t the current is

$$I = I_0 \cos \omega t$$

This current gives a magnetic field inside the solenoid as

$$B = \mu_0 N I$$

and $B = 0$ outside

To calculate the electric field, note that

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

is analogous to $\nabla \times B = \mu_0 J$

so $\partial B/\partial t$ is the source of E.

By symmetry, E must be tangential to a circle of radius r (see figure) and be a function of r only.

Hence

$$\oint_\Gamma \vec{E}\cdot d\vec{l} = 2\pi r E .$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

implies that

$$\oint_\Gamma \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot\hat{n}\, ds$$

$$= -\frac{\partial}{\partial t}\, \pi a^2 \mu_0 N I$$

$$= -\pi a^2 \mu_0 N \frac{dI}{dt}$$

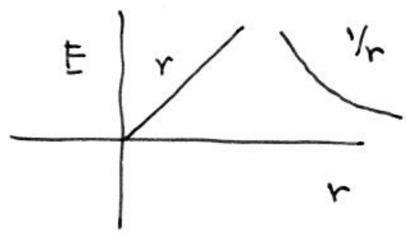
$$= +\mu_0 \pi a^2 N I_0 \omega \sin \omega t$$

(for $r > a$)

$\Rightarrow$

$\Rightarrow$ $$2\pi r E = \mu_0 \pi a^2 N I_0 \omega \sin\omega t$$

$\Rightarrow$ $$E = \frac{\mu_0 a^2 N I_0 \omega \sin\omega t}{2r}$$



For $r < a$

$$E = \mu_0 N I \omega \sin\omega t \,\frac{r^2}{2r}$$

$$= \mu_0 N I \omega \sin\omega t \,\frac{r}{2}$$

2. The induced current in the loop is determined by the induced emf

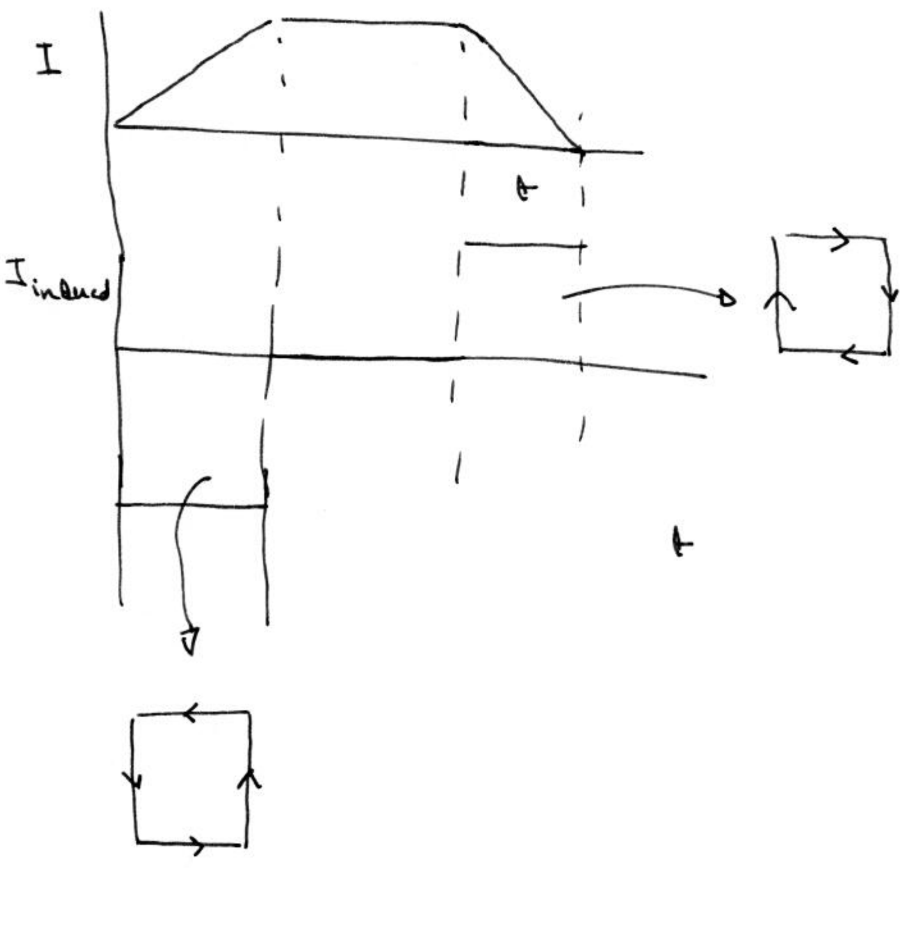
$$\mathcal{E} = -\frac{d\Phi}{dt}$$

The flux Φ is obtaind from the curren I by Biot-savart law; which says

$$\Phi \propto I$$

$$\Rightarrow \quad \mathcal{E} \propto -\frac{dI}{dt}$$

$$\Rightarrow \quad I_{induced} \propto -\frac{dI}{dt}$$



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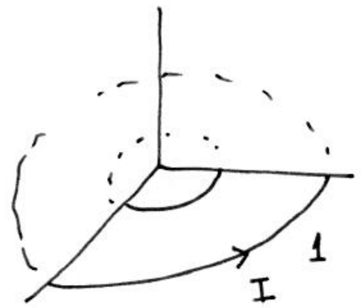
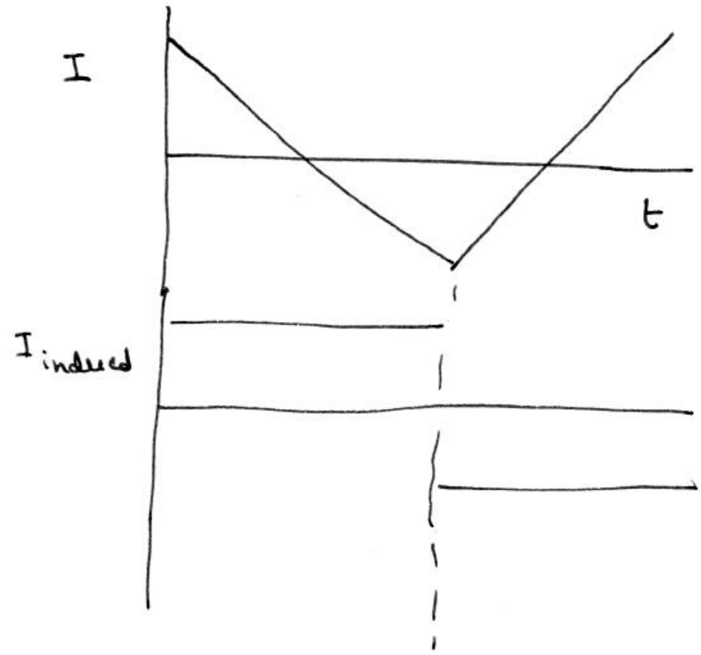
Arguing just like the previous problem:





4.

(a) Maxwell's eqn in free space

$$\nabla \cdot E = 0, \qquad \nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \qquad \nabla \times B = \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

$$\nabla \times B = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & -\frac{E_0}{c} \sin(kx + \omega t) \end{vmatrix}$$

$$= \hat{x} \cdot 0$$

$$+ \hat{y} \frac{E_0}{c} k \cos(kx + \omega t)$$

$$+ \hat{z} \cdot 0$$

$$\frac{\partial E}{\partial t} = \hat{y} E_0 \omega \cos(kx + \omega t)$$

To satisfy Maxwell's 4th eqn.

$$E_0 \frac{k}{c} \cos(kx + \omega t) = \frac{E_0 \omega}{c^2} \cos(kx + \omega t)$$

$$\boxed{c^2 = \frac{1}{\epsilon_0 \mu_0}}$$

$$\Rightarrow \quad \boxed{\cancel{c = k/\omega}} \quad \boxed{c = \frac{\omega}{k}}$$

(b) In free space, speed of light $c = 3 \times 10^8 \, m s^{-1}$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega}$$

$$= \frac{2\pi \; 3 \times 10^8 \; m s^{-1}}{10^{10} \frac{1}{s}}$$

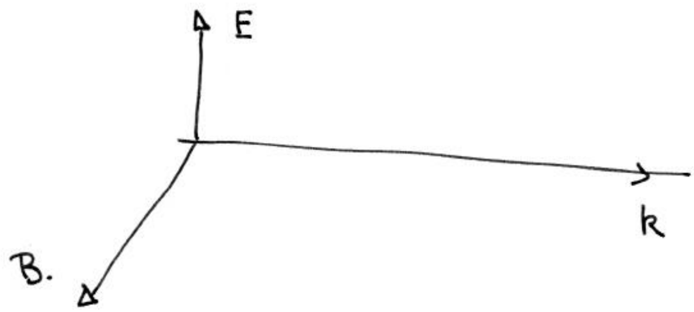
$$= 2\pi \times \frac{3}{100} \; m \sim 0.18 \; m$$

$$\sim 18 \; cm$$

5.

From the solutions of Maxwell's eqns. we know that sunlight is an electromagnetic wave.

As we are far away from the sun it is safe to assume that the wave ~~in~~ ~~a~~ is a plane wave:



As was shown in class (Lecture 7 page 9) The amplitude of E and B are related by

$$B_0 = E_0 / c$$

and
$$\vec{E} = \hat{y}\, E_0 \sin(x - ct)$$
$$\vec{B} = \hat{z}\, B_0 \sin(y - ct)$$

The power is given by the Poynting vector

$$\vec{S} = \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right) = \frac{1}{\mu_0} E_0 B_0 \sin^2(x - ct)\ \hat{x}$$

~~Thus~~
$$= \frac{c}{\mu_0} B_0^2 \sin^2(x - ct)$$

The ~~ave~~ average value of power transported over one period is

$$\frac{1}{T}\int_0^T |\vec{S}|\, dt = \frac{1}{\sqrt{2}} \frac{c B_0^2}{\mu_0}$$

$$1 \frac{\text{kilo watt}}{m^2} = \frac{1}{\sqrt{2}} \frac{3 \times 10^8\, m s^{-1}\; B_0^2}{4\pi \times 10^{-7}\, SI}$$

$$\Rightarrow \quad 10^3 = \frac{3 \times 10^8}{\sqrt{2} \times 4\pi \times 10^{-7}}$$

$$\Rightarrow \quad B_0^2 = \frac{10^3 \times 10^{-7} \times 10^{-8}}{3} \cdot (\sqrt{2} \cdot 4\pi)\, (\text{Tesla})^2$$

$$\simeq 5.6 \times 10^{-12}\, (\text{Tesla})^2$$

$$\Rightarrow \quad \boxed{B_0 \simeq 2 \times 10^{-6}\ \text{Tesla}}$$

Root-mean-square B is $B_0/\sqrt{2} \sim 1.4 \times 10^{-6}$ Tesla