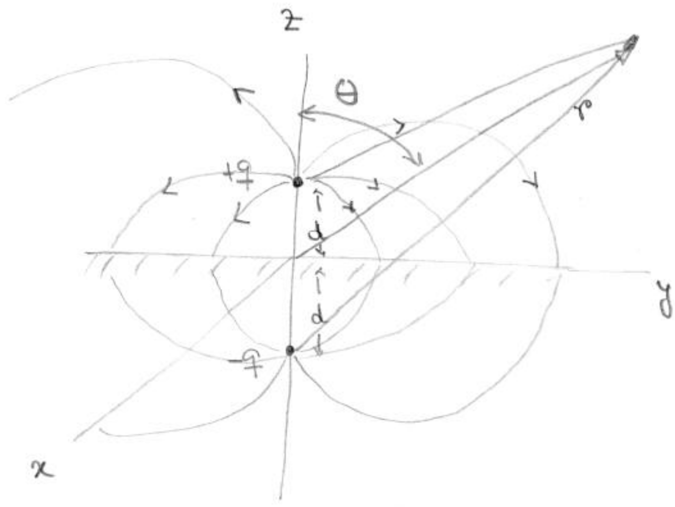
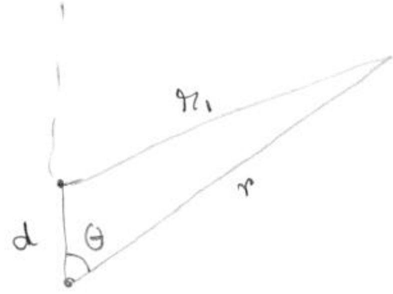
Solution to problem set II





The potential at r, θ is

$$\Phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} - \frac{q}{r_2}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\left(r^2+d^2-2dr\cos\theta\right)^{1/2}} - \frac{1}{\left(r^2+d^2+2dr\cos\theta\right)^{1/2}}\right]$$

Using $z = r\cos\theta$

$r^2 = x^2+y^2+z^2$

we get

$$\Phi(x,y,z) = \frac{q}{4\pi\epsilon_0}\left[\left(x^2+y^2+z^2-2dz+d^2\right)^{-1/2} - \left(x^2+y^2+z^2+2dz+d^2\right)^{-1/2}\right]$$

$$2r \frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

similarly

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial \phi}{\partial x} = \frac{q}{4\pi\epsilon_0} \frac{\partial}{\partial x}\left[\frac{1}{(r^2+d^2-2dz)^{1/2}} - \frac{1}{(r^2+d^2+2dz)^{1/2}} \right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\left(-\frac{1}{\cancel{2}}\right) \frac{1}{(r^2+d^2-2dz)^{3/2}} \cancel{2} r \frac{\partial r}{\partial x} \right.$$

$$\left. + \frac{1}{\cancel{2}} \cdot \frac{1}{(r^2+d^2+2dz)^{3/2}} \cancel{2} r \frac{\partial r}{\partial x} \right]$$

$$= -\frac{q}{4\pi\epsilon_0}\left[\frac{x}{(r^2+d^2-2dz)^{3/2}} - \frac{x}{(r^2+d^2+2dz)^{3/2}} \right]$$

similarly:

$$\frac{\partial \phi}{\partial y} = -\frac{q}{4\pi\epsilon_0}\left[\frac{y}{(r^2+d^2-2dz)^{3/2}} - \frac{y}{(r^2+d^2+2dz)^{3/2}} \right]$$

But

$$\frac{\partial \phi}{\partial z} = \frac{q}{4\pi\epsilon_0}\left[\left(-\frac{1}{\cancel{2}}\right)\frac{1}{(r^2+d^2-2dz)^{3/2}}\left(\cancel{2}r\frac{\partial r}{\partial z} - \cancel{2}d\right)\right.$$

$$\left. + \left(\frac{1}{\cancel{2}}\right)\frac{1}{(r^2+d^2+2dz)^{3/2}}\left(\cancel{2}r\frac{\partial r}{\partial z} + \cancel{2}d\right)\right]$$

$$= -\frac{q}{4\pi\epsilon_0}\left[\frac{z}{(r^2+d^2-2dz)^{3/2}} - \frac{z}{(r^2+d^2+2dz)^{3/2}}\right.$$

$$\left. - \frac{d}{(r^2+d^2-2dz)^{3/2}} - \frac{d}{(r^2+d^2+2dz)^{3/2}}\right]$$

$$\vec{E} = -\left[\hat{x}\frac{\partial \phi}{\partial x} + \hat{y}\frac{\partial \phi}{\partial y} + \hat{z}\frac{\partial \phi}{\partial z}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)\left[\hat{x}\,x + \hat{y}\,y + \hat{z}\,z\right]$$

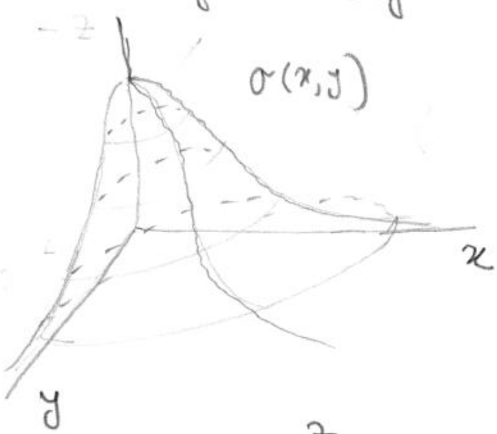
$$- \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1^3} + \frac{1}{r_2^3}\right)\hat{z}\,d$$

At the x-y plane, $z = 0$, $\theta = \frac{\pi}{2}$, $\cos\theta = 0$

$$\Rightarrow r_1 = r_2 = (x^2+y^2+d^2)^{1/2}$$

$$\Rightarrow \vec{E}(x,y,0) = -\frac{q}{4\pi\epsilon_0}\frac{2d\,\hat{z}}{(x^2+y^2+d^2)^{3/2}}$$

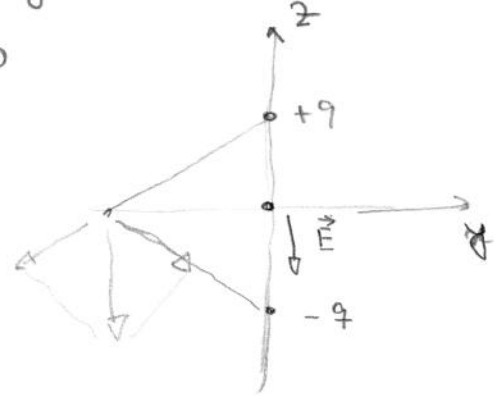
By Gauss's theorem, the surface charge density



$$\sigma(x,y) = \epsilon_0 \vec{E} \cdot \hat{z}$$

$$= -\frac{q}{4\pi} \frac{2d}{\left(x^2+y^2+d^2\right)^{3/2}}$$

At the origin, $x=0$, $y=0$



$$\vec{E} = -\frac{q}{4\pi\epsilon_0} \frac{2d}{d^3} \hat{z}$$

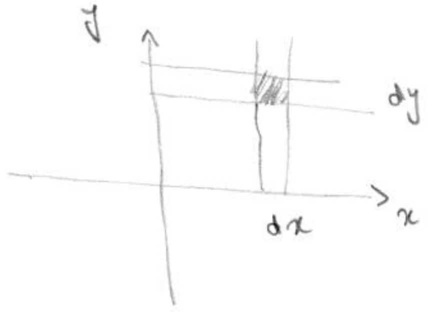
$$= -\frac{1}{4\pi\epsilon_0} \frac{2q}{d^2} \hat{z}$$

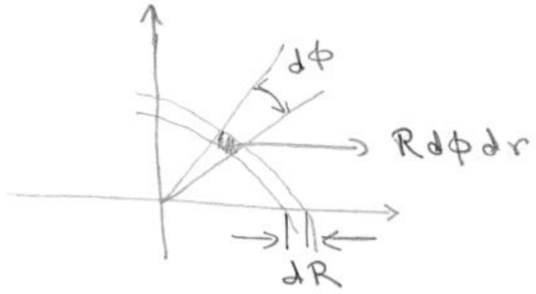
which is exactly the same expression obtained in the first problem of Problem set I

The total induced charge

$$q_{in} = -\frac{q}{4\pi} 2d \int_{-\infty}^{+\infty} \frac{dx\,dy}{\left(x^2+y^2+d^2\right)^{3/2}}$$

This integration is best done in plain polar coordinate





we substitute $x = R\cos\phi$

$y = R\sin\phi$

$$x^2 + y^2 = R^2$$

$$dx\,dy = R\,d\phi\,dr$$

$$q_{in} = -\frac{q}{4\pi}\,2d\int_{R=0}^{\infty}\int_{\phi=0}^{2\pi}\frac{R\,d\phi\,dR}{\left(R^2+d^2\right)^{3/2}}$$

$$= -\frac{q}{\cancel{4\pi}}\left(\cancel{2}d\right)\left(\cancel{2\pi}\right)\int_0^{\infty}\frac{R\,dR}{\left(R^2+d^2\right)^{3/2}}$$

substitute $R^2 + d^2 = \xi^2$, $\quad 2R\,dR = 2\xi\,d\xi$

$$q_{in} = -q\,d\int_d^{\infty}\frac{\cancel{\xi}\,d\xi}{\xi^{\cancel{3}2}}$$

$$= +q\,d\,\frac{1}{\xi}\Big|_d^{\infty} = -q$$

The total induced charge is exactly equal to the image charge. which we could guess without any calculation.