· In free space:

$$\nabla \cdot E = 0 \qquad \nabla \times B = \epsilon_0 \mu_0 \frac{\partial E}{\partial t}$$

$$\nabla \cdot B = 0 \qquad \nabla \times E = -\frac{\partial B}{\partial t}$$

The equations are symmetric in $\vec{E}$ and $\vec{B}$.

A particularly interesting consequence is the following

$$\nabla \times \nabla \times B = \nabla(\nabla \cdot B) - \nabla^2 B$$

$$= -\nabla^2 B \quad (\text{because } \nabla \cdot B = 0)$$

$$\nabla \times \epsilon_0 \mu_0 \frac{\partial E}{\partial t} = \epsilon_0 \mu_0 \frac{\partial}{\partial t} \nabla \times E$$

$$= -\epsilon_0 \mu_0 \frac{\partial^2 B}{\partial t^2}$$

putting together

$$\boxed{\nabla^2 B = \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2}}$$

which is the wave equation. You can check that $\vec{E}$ satisfies the same eqn. Thus, in free space Maxwell's equations give electromagnetic waves! Which travels with the speed of light.

Notes on waves

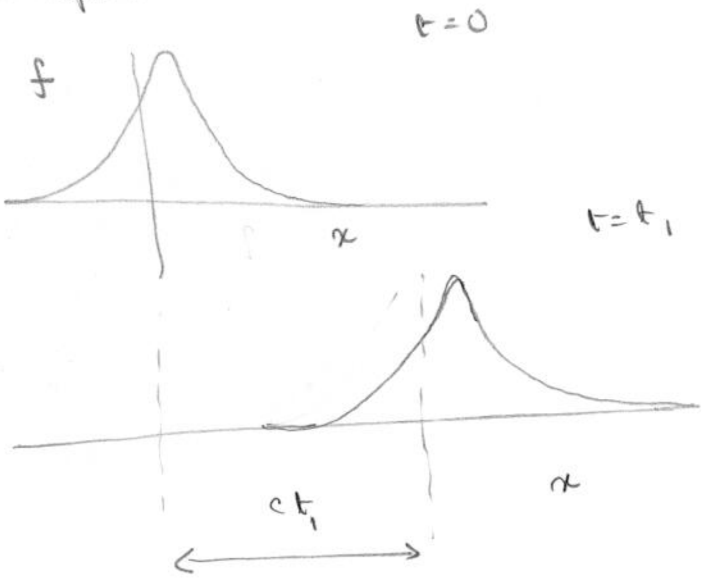
Intutively a wave is a pattern that travels with a certain speed. For example consider the function

$$f(x - ct)$$

at $t = 0$, $x = 0$, we obtain $f(0)$

at $t = t_1$, $x = ct_1$ we obtain the same value $f(0)$.

$\Rightarrow$ whatever the function f is it travels with speed c.



what kind of equation does $f(x-ct)$ satisfies?

clearly $\dfrac{\partial f}{\partial x} = f'$ $\qquad \dfrac{\partial f}{\partial t} = -c f'$

$$\frac{\partial^2 f}{\partial x^2} = f'' \qquad \frac{\partial^2 f}{\partial t^2} = +c^2 f''$$

$$\Rightarrow \quad \frac{\partial f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$$

what we wrote down for the magnetic field is the three dimensional version of the same equation.

$$\nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

As a concrete example I propose that the following is a solution of the Maxwell's eqn in free space:

$$\vec{E} = \hat{x}\, E_0 \sin(x - ct)$$

$$\vec{B} = \hat{z}\, B_0 \sin(y - ct)$$

Let us check if this works:

$$\frac{\partial \vec{E}}{\partial t} = \hat{x}\, E_0 \cos\left((y - ct)\right)(-c)$$

$$\nabla \times B = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & B_0 \sin(y - ct) \end{vmatrix}$$

$$= \hat{x}\, B_0 \cos(y - ct)$$

Substituting in Maxwell's law

$$\nabla \times B = \hat{z}\ B_0 \cos(y - ct)$$

$$\frac{\partial E}{\partial t} = \hat{x}\ E_0 (-c) \cos(y - ct)$$

$$\nabla \times \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

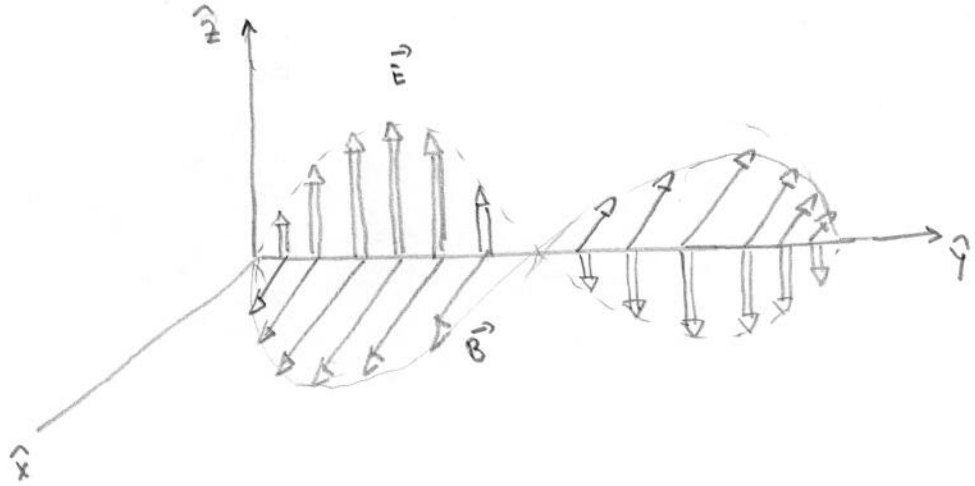
we obtain: $B_0 = (\epsilon_0 \mu_0)\, c\, E_0$

$$= \frac{c}{c^2} E_0$$

$$\Rightarrow \quad \boxed{B_0 = E_0/c}$$

So the proposed solutions are solutions of Maxwell's eqns with $B_0 = E_0/c$

It is a sine wave that travels along $\hat{y}$ direction with $\vec{E}$ and $\vec{B}$ and $\hat{x}$ and $\hat{z}$ direction



The $\vec{E}$, the $\vec{B}$ and the direction of propagation $\hat{y}$ makes a triad.

How is energy transported ?

So far we have associated energy with charges and currents. For example, for electrostatic energy of a collection of point charges we wrote

$$U = \frac{1}{4\pi\epsilon_0} \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{r_{ij}}$$ (Lecture 2, section 2.3)

(because every pair of charges is counted twice)

$$= \frac{1}{2} \sum_i q_i \sum_j \frac{q_j}{r_{ij}} \frac{1}{4\pi\epsilon_0}$$

$$= \frac{1}{2} \sum_i q_i \phi_{ij}$$ ← potential at i due to all charges other than i

For a continuous charge distribution :

$$U = \frac{1}{2} \int_V \rho\, dV\, \phi$$

Now let us try some mathematical juglary

$$U = \frac{1}{2}\int \rho \phi \, dV$$

$$= \frac{1}{2} \epsilon_0 \int (\nabla \cdot E) \, \phi \, dV$$

Gauss's law

$$= \frac{1}{2}\int \left[\nabla \cdot (\phi \vec{E}) \epsilon_0 - \vec{E} \cdot (\nabla \phi) \right] dV$$

$$\nabla \cdot (\phi \vec{E}) = \phi (\nabla \cdot \vec{E}) + (\vec{\nabla} \phi) \cdot \vec{E}$$

$$= \frac{\epsilon_0}{2}\int_V E^2 \, dV + \frac{\epsilon_0}{2}\int_V \nabla \cdot (\phi \vec{E}) \, dV \qquad \left[\because \vec{E} = -\nabla \phi \right.$$

$$= \frac{\epsilon_0}{2}\int_V E^2 \, dV$$

Because the second term is zero.

We show this by showing that

$$\int_V \nabla \cdot (\phi \vec{E}) \, dV = \oint_S (\phi \vec{E}) \cdot \hat{n} \, ds$$

← Gauss's flux theorem

The volume integral includes all volume, so the surface of the surface integral is at infinity. There $\vec{E} = 0$, $\phi = 0$, hence the surface integral is zero.

To conclude

$$U = \frac{\epsilon_0}{2} \int_V \underline{E}^2 \, dV$$

Here we can imagine that instead of the energy being stored in the charges, it is stored in the electric field.

For the case of electric and magnetic field the energy would be

$$U = \frac{1}{2} \int_V \left(\epsilon_0 \underline{E}^2 + \frac{1}{\mu_0} \underline{B}^2 \right) dV$$

Hence we can use the concept of energy density.

A small volume δV stores energy $\mathcal{E} \, \delta V$ where

$$\mathcal{E} = \frac{1}{2} \left(\epsilon_0 \underline{E}^2 + \frac{1}{\mu_0} \underline{B}^2 \right)$$

with, total energy stored in all space, to be

$$U = \int \mathcal{E} \, dV$$

If energy is conserved then it should obey the same conservation law as electric charge.

$$\Rightarrow \quad \partial_t \mathcal{E} + \nabla \cdot \vec{j}_{\mathcal{E}} = 0$$

where $\vec{\dot{J}}_\varepsilon$ is the vector denoting the flux of energy.

what would $\vec{\dot{J}}_\varepsilon$ look like?

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial}{\partial t} \frac{1}{2}\left(\epsilon_0 \underline{E}^2 + \frac{1}{\mu_0} B^2\right)$$

$$= \left(\epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \frac{1}{\mu_0} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t}\right)$$

$$\epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \epsilon_0 \vec{E} \cdot \frac{1}{\epsilon_0 \mu_0} \nabla \times B = \frac{1}{\mu_0} \vec{E} \cdot (\vec{\nabla} \times \vec{B})$$

$$\frac{1}{\mu_0} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} = -\frac{1}{\mu_0} \vec{B} \cdot (\vec{\nabla} \times \vec{E}) = -\frac{1}{\mu_0} \vec{B} \cdot (\vec{\nabla} \times \vec{E})$$

putting together

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{\mu_0}\left[\vec{E} \cdot (\vec{\nabla} \times \vec{B}) - \vec{B} \cdot (\vec{\nabla} \times \vec{\underline{E}})\right]$$

$$= -\frac{1}{\mu_0} \nabla \cdot (\vec{E} \times \vec{B})$$

$$\Rightarrow \quad \vec{\dot{J}}_\varepsilon = \frac{1}{\mu_0}\left(\vec{E} \times \vec{B}\right) \equiv \vec{S}$$

Known as the Poynting vector, which gives the flux of energy. The direction of $\vec{S}$ is the direction of propagation of light; which shows the $\vec{E}$, $\vec{B}$ and $\vec{S}$ forms a triad.