8. Maxwell's Equations

Let us write down the laws of electromagnetism the way we know them now,

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

And just for the fun of it calculate

$$\nabla\cdot(\nabla\times E) = -\nabla\cdot\frac{\partial\vec{B}}{\partial t}$$

by a mathematical theorem $= 0$

because $\nabla\cdot B = 0$ an experimental law of nature. $0 =$

And everything is consistent.

Now, let us try the same game on Ampere's law:

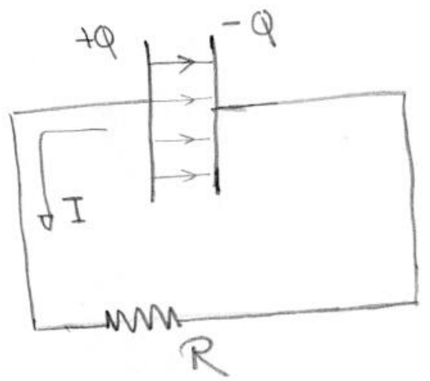
$$\nabla\cdot(\vec{\nabla}\times\vec{B}) = 0 \quad \text{(a mathematical theorem)}$$

$$\nabla\cdot\mu_0\vec{J} = \mu_0\,\vec{\nabla}\cdot\vec{J} = -\mu_0\frac{\partial\rho}{\partial t}$$

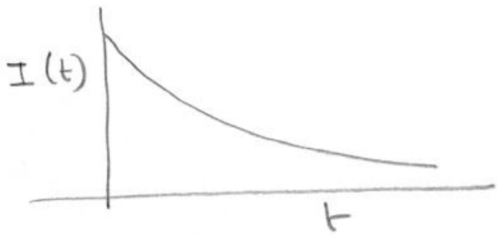
$\neq 0$ in general, only true in statics.

we obtain clear inconsistancy! Something must be wrong with Ampere's law when charges move.

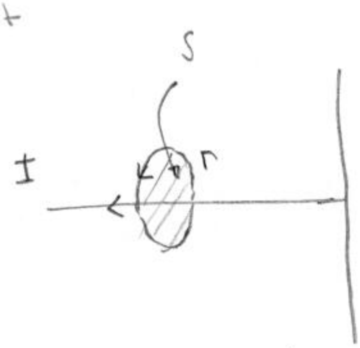
Another, more physical, way to see the same problem is to consider the discharging of a capacitor



This sets of a current which will go to zero, and so will the charge

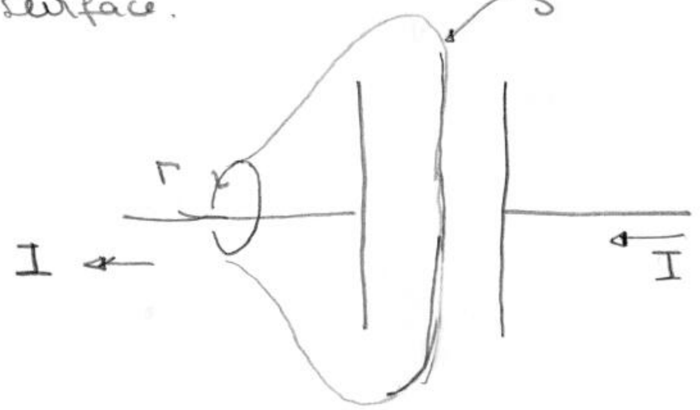


Now consider the magnetic field set up by this current



Ampere's law is valid for the loop Γ and the surface S.

Now consider the same loop, but a different surface.



Clearly no charges are flowing through the surface S'

Hence current piercing this surface is zero

but the magnetic field calculated on the loop Γ remains the same! So Ampere's law is not valid.

This situation cannot happen in the "static" case.

Maxwell proposed the following idea to "fix" Ampere's law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \vec{J}_d$$

displacement current.

$$\nabla \cdot \nabla \times B = \mu_0 \, \nabla \cdot J + \mu_0 \, \nabla \cdot \vec{J}_d$$

$$= \mu_0 \frac{\partial \rho}{\partial t} + \mu_0 \, \nabla \cdot J_d$$

$$= - \mu_0 \epsilon_0 \nabla \cdot \frac{\partial \vec{E}}{\partial t} + \mu_0 \nabla \cdot \vec{J}_d$$

we have used
Gauss's law
$\nabla \cdot E = \rho / \epsilon_0$

So, if we choose $\vec{J}_d = \epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

the problem is fixed, we obtain

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}$$

Comments

- dimensions: $\nabla \times E = -\dfrac{\partial B}{\partial t}$

$$\Rightarrow \quad \frac{E}{\text{length}} = \frac{B}{\text{time}}$$

$$\nabla \times B = \frac{B}{\text{length}} = \frac{\text{time } E}{\text{length}^2}$$

$$= \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

$$= (\mu_0 \epsilon_0) \frac{E}{\text{time}}$$

comparing $\mu_0 \epsilon_0 = \dfrac{\text{time}^2}{\text{length}^2}$

$$\Rightarrow \quad \frac{1}{\mu_0 \epsilon_0} = \frac{1}{\text{velocity}^2}$$

- Numbers $\dfrac{\mu_0}{4\pi} = 10^{-7}$ SI

$$\epsilon_0 = 8{\cdot}85 \times 10^{-12} \text{ SI}$$

$$\frac{1}{\mu_0 \epsilon_0} = \frac{}{8{\cdot}85 \times 10^{-12} \times 4\pi \times 10^{-7}} (m s^{-1})^2$$

$$= \frac{1}{8{\cdot}85 \times 4\pi} \times 10^{+19} (m s^{-1})^2$$

$$\left(\frac{1}{\mu_0 \epsilon_0}\right)^{1/2} = \left(\frac{10^3}{8{\cdot}85 \times 4\pi}\right)^{1/2} 10^8 \, m s^{-1} = 3 \times 10^8 \, m s^{-1}$$

— speed of light!