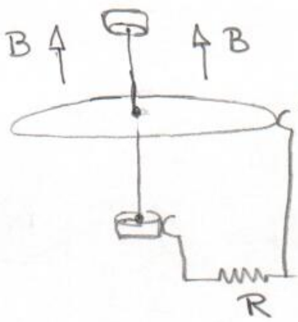
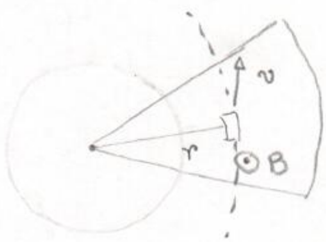
Induction

6.4 Electromotive force from Faraday's law.

Example 6.3



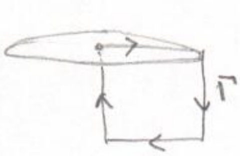
The disk above is rotating with an angular velocity ω, calculate the induced EMF.



$$\vec{v} = \omega r \, \hat{e}_\phi$$

$$\vec{v} \times \vec{B} = \omega B r \, \hat{e}_r$$

The emf $\quad \mathcal{E} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$



$$= \int_0^a \omega B r \, dr$$

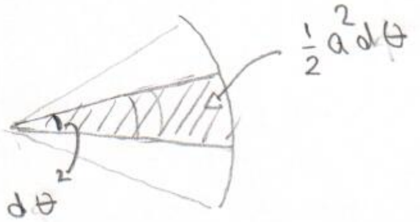
$$= \omega B \frac{a^2}{2}$$

This we obtain by the Lorentz force law.

But now apply Faraday's law with a little creativity:

The net flux through the disk

$$\Phi = \pi a^2 B$$

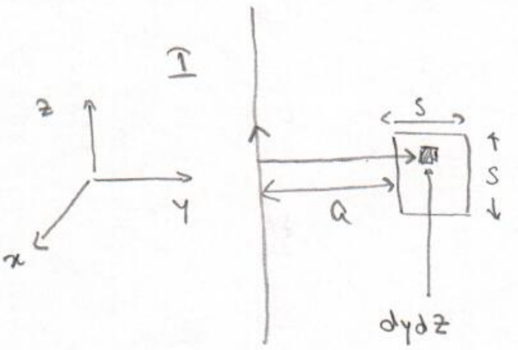


Consider a small segment of the disk. In time dt the "flux lines" cut by this area: $B\,\frac{1}{2}a^2\frac{d\theta}{dt} = \frac{1}{2}a^2\omega B$

which is exactly the EMF.

The cleanest interpretation, in this case, is the Lorentz force one.

Example 6.4



(a) what is the flux through the loop:

$$\Phi = \int_{\text{loop}} B(y,z)\,dy\,dz$$

$$B(y)\,2\pi y = \mu_0 I$$

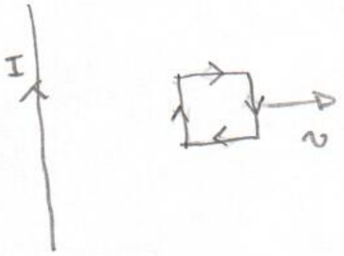
$$\Rightarrow \quad B(y) = \frac{\mu_0 I}{2\pi y}$$

$$\Phi = \int_{\text{loop}} \frac{\mu_0 I}{2\pi y}\,dy\,dz$$

$$= \frac{\mu_0 I}{2\pi}\, s. \int_a^{a+s} \frac{dy}{y} = \frac{\mu_0 I}{2\pi}\, s. \left[\ln(s+a) - \ln a\right]$$

$$= \frac{\mu_0 I s}{2\pi} \ln\left(1 + \frac{s}{a}\right)$$

(b) Pull the loop with velocity v. What is the induced EMF

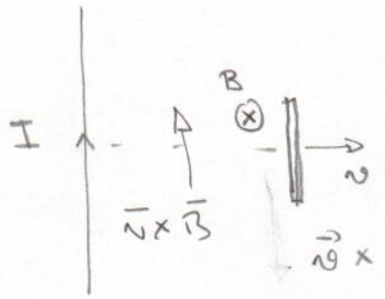


$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$= -\frac{\mu_0 I s}{2\pi}\frac{1}{\left(1+\frac{s}{a}\right)}\left(-\frac{s}{a^2}\right)\frac{da}{dt}$$

$$= \frac{\mu_0 I s^2}{2\pi a^2}\frac{1}{\left(1+s/a\right)} v$$

(c) Apply Lorentz force law:



$$\int_{left} d\vec{l}\cdot\left(\vec{v}\times\vec{B}\right) = \frac{\mu_0 I}{2\pi a} s.v$$

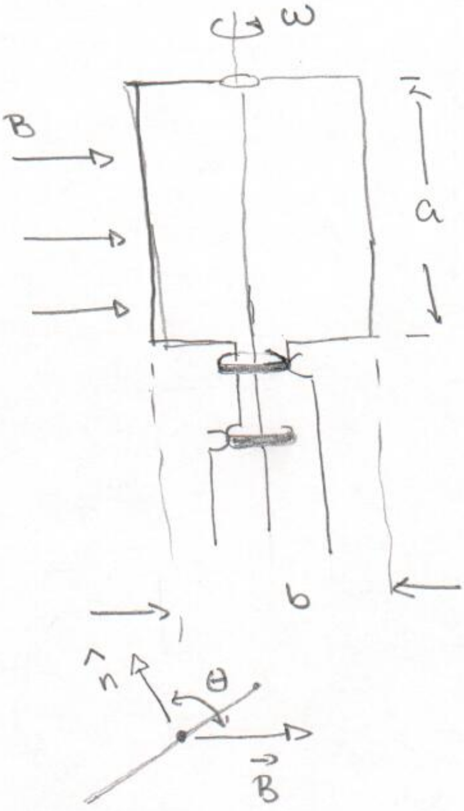
$$\int_{right} d\vec{l}\cdot\left(\vec{v}\times\vec{B}\right) = -\frac{\mu_0 I}{2\pi(a+s)} s.v$$

The difference

$$= \frac{\mu_0 I s}{2\pi}\left[\frac{1}{a} - \frac{1}{(a+s)}\right] v$$

$$= \frac{\mu_0 I s^2 v}{2\pi a(a+s)}$$

Example 6.5



$\theta = \omega t$

what is the induced emf?

$$\Phi = \int \vec{B} \cdot \hat{n} \, ds$$

$$= B(ab) \cos\theta$$

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

$$= -B(ab) \sin\theta \frac{d\theta}{dt}$$

$$= -B(ab) \sin\theta \; \omega.$$

$$= -B(ab)\, \omega \sin(\omega t)$$

6.5 comments on Faraday's law.

- In principle we could define a new field $\vec{G}$ such that

$$\vec{\nabla} \times \vec{G} = -\frac{\partial \vec{B}}{\partial t}$$

and the force on a charge q due to $\vec{G}$ would be $\vec{F} = q\vec{G}$

The equations of electrodynamics (so far) would then look like

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \cdot \vec{G} = 0 \quad \nabla \times \vec{G} = -\frac{\partial \vec{B}}{\partial t}$$

and $$\vec{F} = q\left(\vec{E} + \vec{G} + \vec{v} \times \vec{B}\right)$$

It is certainly convenient to consider $\vec{E}$ and $\vec{G}$ together as electric field as they act on charges in exactly the same way, although the sources are different.

- $$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \nabla \times \vec{B} = \mu_0 \vec{J}$$

By comparison:

$$\vec{E} = \frac{1}{4\pi} \int \frac{\left(\partial \vec{B}/\partial t\right) \times \hat{r}}{r^2} \, dv$$

can always be calculated by using Ampere's law if enough symmetry exists.

- $$\vec{B} = \nabla \times \vec{A},$$

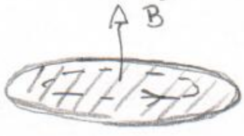
$$\begin{aligned} \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ &= -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} \\ &= -\vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} \end{aligned}$$

$$\Rightarrow \boxed{\vec{E} = -\frac{\partial \vec{A}}{\partial t} + \vec{\nabla} \phi}$$

electrostatic potential

constant of integration

Example 6·6



$\vec{B}$ fills the region shown, and changes with time in the following way

$$B = B_0 \cos\omega t$$

Find the electric field thus induced.

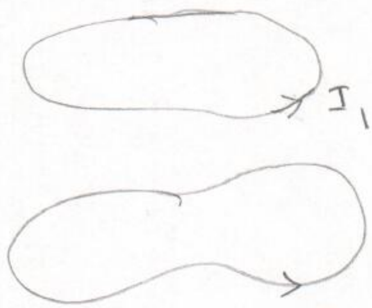
$$\oint \vec{E} \cdot d\vec{l} = 2\pi r E = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \hat{n}\, ds$$

$$= -\pi r^2 B_0 \sin(\omega t)\, \omega$$

$$\Rightarrow \quad E = -\frac{r}{2} \omega B_0 \sin \omega t$$

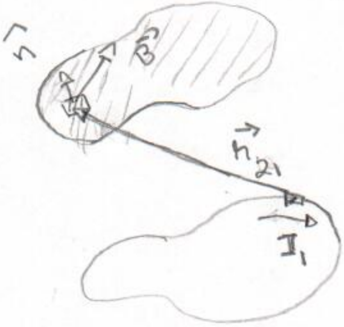
6·6 Inductance:

Consider two loops of current. clearly, if I change I_1, the magnetic field at loop 2 due to I_1 changes. This must change the flux through loop 2, Φ_{21} due to loop 1. This must set up a new emf at loop 2, $\mathcal{E}_{21}$ due to loop 1, and hence the current in loop 2 will also change.



Let us try to study this problem. How do we get the flux at loop 2 due to curren I_1?

Obviously, Biot - Savart law gives me:



$$\vec{B}_{21} = \frac{\mu_0}{4\pi} \oint \frac{I_1 \, d\vec{l}_1 \times \hat{r}_{12}}{r_{21}^2}$$

Then I calculate the flux

$$\Phi_{21} = \int_{S_2} \vec{B}_{21} \cdot \hat{n}_2 \, ds_2$$

But Biot - savart law does not apply here because it's for static current and I_1 is not static! But if I_1 is not changing very fast (we shall see in the next lecture how fast) the $\vec{B}_{21}$ depends (via Biot-Savart) on the instantanious I_1. So we can proceed with the rest of the calculations.

It is actually easier to perform the actual calculation using the vector potential.

$$\mathcal{E}_{21} = - \frac{d}{dt} \int \vec{B}_{21} \cdot \hat{n}_2 \, ds$$

$$= - \frac{d}{dt} \oint_{\Gamma_2} \vec{A}_{21} \cdot d\vec{l}_2$$

$$\vec{A}_{21} = \frac{\mu_0}{4\pi} \oint_{\Gamma_1} \frac{I_1 \, d\vec{l}_1}{r_{12}}$$

$$\mathcal{E}_{21} = - \frac{\mu_0}{4\pi} \frac{d}{dt} \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{I_1 \left(d\vec{l}_1 \cdot d\vec{l}_2 \right)}{r_{21}}$$

The only quantity that varies with time is I_1, if we keep both the circuits fixed.

$$\mathcal{E}_{21} = M_{21} \frac{dI_1}{dt}$$

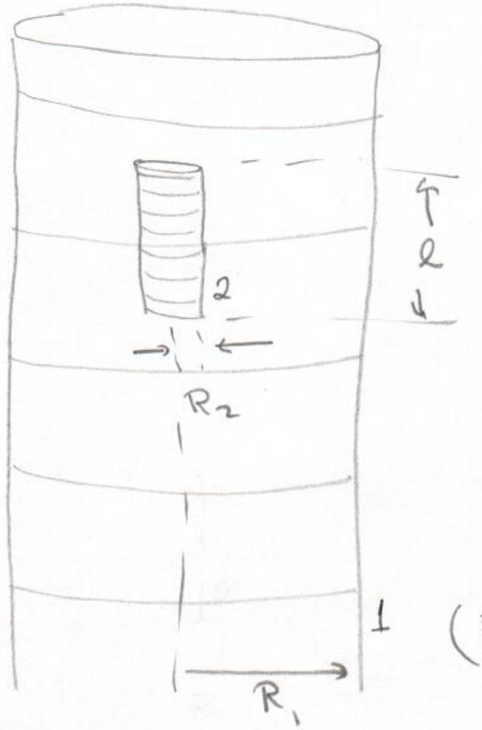
where M_{21} depends only on the geometry of the two circuits:

$$M_{21} = - \frac{\mu_0}{4\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r_{21}}$$

Clearly $M_{12} = M_{21} \equiv M$

This we call the mutual inductance of the circuits. It is always negative.

Example 6.7



A short solenoid, inside a long solenoid. What is the mutual inductance?

(Field at 2 due to 1)

$$= B_{21} = \mu_0 N_1 I_1$$

(Flux through 2)

$$= \Phi_{21} = B_{21} \pi R_2^2 N_2 l.$$

$$= \mu_0 (\pi R_2^2) N_1 I_1 N_2 l$$

$$\Rightarrow M_{21} = \mu_0 (N_1 N_2) \pi l R_2^2$$

The $M_{12} = M_{21}$ would have been very difficult to calculate in the other way!

6.7 Self inductance:

Clearly, when a current builds up in a circuit the flux enclosed by the circuit itself is changing. This would set up an additional emf which should be given by

$$\mathcal{E} = -\mathcal{L}\frac{dI}{dt}$$

Lenz's law.

This is called the self inductance.

Putting both the inductances together, if we have two current loop 1 and 2.

$$\mathcal{E}_1 = M_{11}\frac{dI_1}{dt} + M_{12}\frac{dI_2}{dt}$$

$$\mathcal{E}_2 = M_{21}\frac{dI_1}{dt} + M_{22}\frac{dI_2}{dt}$$

where $M_{11} = -\mathcal{L}_1$

$M_{22} = -\mathcal{L}_2$

$M_{12} = M_{21} = M$

units :

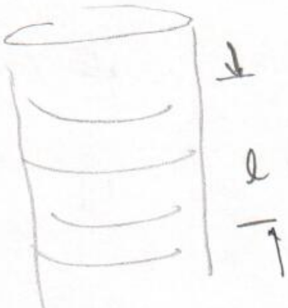
I : ampere

$\mathcal{E}$: volt

$\mathcal{L}$; M : Henries.

Example 6.8

self-inductance of a solenoid:



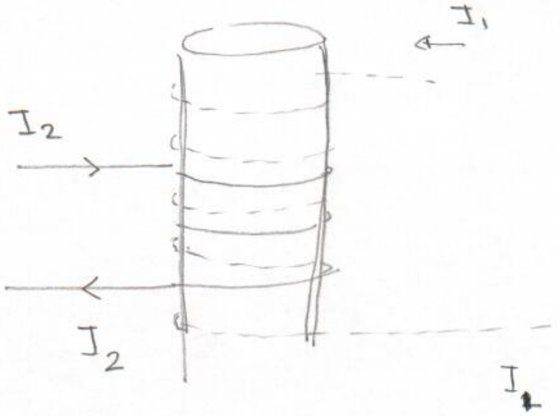
The magnetic field

$$B = \mu_0 N I$$

$$\underbrace{\Phi}_{\text{for length } l} = \mu_0 I N \pi R^2 \underbrace{(N l)}_{\text{no. of coils in length } l}$$

$$\boxed{L = \mu_0 \left(\pi R^2 l\right) N^2}$$

Example 6.9



To solenoids are wound on the same cylinder. We send current $I_1 \cos \omega t$ in one of them. What is the emf through the other one?

The magnetic field of 1

$$B = \mu_0 N_1 I_1$$

The flux in 1 due to 1 itself

$$\Phi_{11} = \mu_0 (\pi R^2 l) N_1^2$$

$$\Rightarrow \quad \mathcal{E}_{11} = \mu_0 \pi R^2 l N_1^2 \frac{dI_1}{dt}$$

$$\Phi_{21} = \mu_0 (\pi R^2 l) N_1 N_2$$

$$\mathcal{E}_{21} = \mu_0 \pi R^2 l N_1 N_2 \frac{dI_1}{dt}$$

$$\frac{\mathcal{E}_{21}}{\mathcal{E}_{11}} = \frac{N_2}{N_1}$$

Can be used as a step-up or step-down transformer.

Magnetic energy

The force on a charge q is

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

If in an electric field we move a charge from one point to another, the work done is,

$$W = \int_A^B \vec{F} \cdot d\vec{l}$$

$$= q \int_A^B \vec{E} \cdot d\vec{l}$$

Now calculate the work done in taking a charge on a circular path

$$W = q \oint \vec{E} \cdot d\vec{l}$$

If $\vec{E}$ is only an electrostatic field the clearly

$$W = q \oint (\vec{\nabla} \phi) \cdot d\vec{l} = 0$$

But in general:

$$\vec{E} = \frac{\partial \vec{A}}{\partial t} + \nabla \phi$$

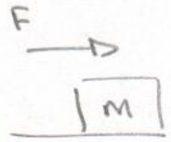
$$W = q \frac{\partial}{\partial t} \oint_\Gamma \vec{A} \cdot d\vec{l} \neq 0$$

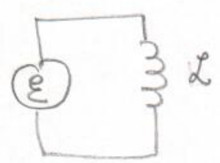
If we take an unit charge along an electric circuit, the work done is

$$\oint_{\text{circuit}} \vec{E} \cdot \vec{dl} = \mathcal{E}$$

But this work is being done only when the current increases from 0 to I. After which no work is being done.

Think of the following analogy.





$$F = m\frac{dv}{dt} \qquad\qquad \mathcal{E} = -\mathcal{L}\frac{dI}{dt}$$

The voltage $V = \mathcal{L}\dfrac{dI}{dt}$

$$F = m\frac{dv}{dt} \qquad\qquad V = \mathcal{L}\frac{dI}{dt}$$

F (force)	V (voltage)
v (velocity)	I (current)
x (displacement)	q (charge)
mv (momentum)	$\mathcal{L} I$
$\frac{1}{2}mv^2$ (kinetic energy)	$\frac{1}{2}\mathcal{L} I^2$ ← magnetic energy.

Note that, the flux of magnetic field through its own circuit is

$$\Phi = \mathcal{L} I$$

$$= \int_S \vec{B} \cdot \hat{n}\, ds$$

$$= \oint_\Gamma \vec{A} \cdot d\vec{l}$$

Energy stored

$$= \frac{1}{2} \mathcal{L} I^2$$

$$= \frac{1}{2} \mathcal{L} I \cdot I$$

$$= \frac{1}{2} I \oint \vec{A} \cdot d\vec{l}$$

$$= \frac{1}{2} \oint \left(\vec{A} \cdot \vec{I} \right) dl$$

$$= \frac{1}{2} \int \vec{A} \cdot \vec{J}\, dV$$ for volume currents

$$= \frac{1}{2\mu_0} \int \vec{A} \cdot \left(\nabla \times \vec{B} \right) dV$$

$$= \frac{1}{2\mu_0} \int \left[B^2 - \nabla \cdot \left(\vec{A} \times \vec{B} \right) \right] dV$$

$\Rightarrow$ Energy store

$$U = \frac{1}{\mu_0}\int B^2\, dV - \int (\vec{A} \times \vec{B})\, dV$$

$$= \frac{1}{\mu_0}\int \frac{B^2}{2}\, dV - \oint_S (\vec{A} \times \vec{B}) \cdot \hat{n}\, ds$$

$\downarrow$

zero if the surface is far away

$$\Rightarrow \boxed{U = \frac{1}{\mu_0}\int \frac{B^2}{2}\, dV}$$