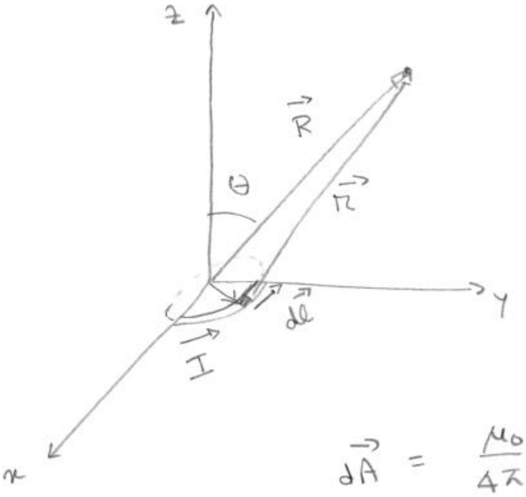
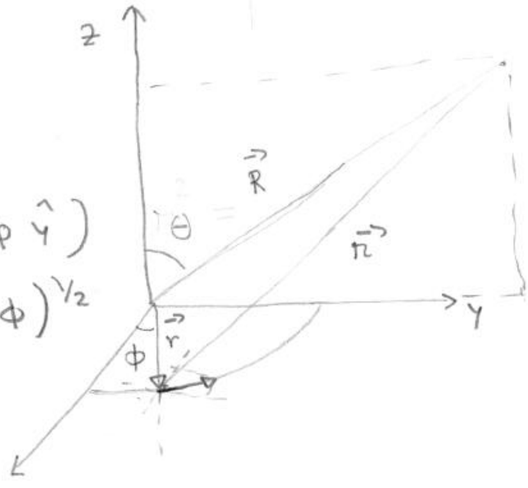
5·5

The magnetic dipole



Let us calculate the vector potential of a current loop at a point $\vec{R}$ from its center.



$$\vec{dA} = \frac{\mu_0 I}{4\pi} \frac{\vec{dl}}{\pi}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{d\phi\, r\left(-\sin\phi\, \hat{x} + \cos\phi\, \hat{y}\right)}{\left(r^2 + R^2 - 2Rr\sin\theta\sin\phi\right)^{1/2}}$$

$$\int_0^{2\pi} \frac{\cos\phi\, d\phi}{\left(P - Q\sin\phi\right)^{1/2}} = 0$$

So the vector potential has only the x component; which is

$$-\frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{\sin\phi\, d\phi}{\left(r^2 + R^2 - 2Rr\sin\theta\sin\phi\right)^{1/2}}$$

which in its full glory can look quite complex.

$$\vec{R} = \left(\hat{z}\cos\theta + \hat{y}\sin\theta\right) R$$

$$\vec{r} = \left(\hat{x}\cos\phi + \hat{y}\sin\phi\right) r$$

$$\vec{\pi} = \vec{R} - \vec{r}$$

$$= -\hat{x}\cos\phi\, r + \hat{y}(R\sin\theta - r\sin\phi) + \hat{z}\, R\cos\theta$$

$$\pi^2 = r^2\cos^2\phi + R^2\cos^2\theta + R^2\sin^2\theta + r^2\sin^2\phi - 2Rr\sin\theta\sin\phi$$

$$= r^2 + R^2 - 2Rr\sin\theta\sin\phi$$

$$\vec{dl} = r\, d\phi\, \hat{e}_\phi$$

$$= r\, d\phi\left(-\sin\phi\, \hat{x} + \cos\phi\, \hat{y}\right)$$

But the main purpose of this exercise is to look at the potential for large R

In that approximation:

$$\frac{1}{\left(R^2 + r^2 - 2rR\sin\theta\sin\phi\right)^{1/2}}$$

$$= \frac{1}{R}\left(1 + \xi^2 - 2\xi\sin\theta\sin\phi\right)^{-1/2} , \quad \xi = \frac{r}{R}$$

$$= \frac{1}{R}\left[1 + \xi\sin\theta\sin\phi + O(\xi^2)\right]$$

terms of order ξ^2 and higher which will be ignored.

$$\Rightarrow \vec{A}(0, R\sin\theta, R\cos\theta)$$

$$= -\hat{x}\frac{\mu_0 I}{4\pi}\frac{r}{R}\int_0^{2\pi} d\phi \, \sin\phi\left(1 + \frac{r}{R}\sin\theta\sin\phi\right)$$

$$= -\hat{x}\frac{\mu_0 I}{4\pi}\frac{r}{R}\left(0 + \frac{r}{R}\sin\theta\,\pi\right) \qquad \int_0^{2\pi}\sin^2\phi\,d\phi = \frac{1}{2}2\pi$$

$$= -\hat{x}\frac{\mu_0}{4\pi}\frac{I\pi r^2}{R^2}\sin\theta$$

$$= -\hat{x}\frac{\mu_0}{4\pi}\frac{m\sin\theta}{R^2}$$

$m \equiv$ magnetic dipole moment $= \pi r^2 I$

There is a nicer way to write this result,

define $\vec{m} = I \pi r^2 \hat{z}$ (by giving the area of the loop a direction)

$= m \hat{z}$

Evaluate $\vec{m} \times \hat{R}$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & m \\ 0 & \sin\theta & \cos\theta \end{vmatrix}$$

$= - \hat{x} \; m \sin\theta$

$$\Rightarrow \boxed{\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{R}}{R^2}} \qquad m = I \pi r^2 \hat{n}$$

vector potential of a magnetic dipole.

The field of a dipole can be calculated from this expression

$\vec{B} = \vec{\nabla} \times \vec{A}$

$$\vec{\nabla} \times \vec{A} = \frac{\mu_0}{4\pi} (\vec{m} \cdot \vec{\nabla}) \left(\frac{\hat{R}}{R^2} \right)$$

5.6

The Feynman trick to calculate

$$\vec{\nabla} \times (\vec{F} \times \vec{G})$$

$\vec{\nabla}$ as a differential operator acts on both $\vec{A}$ and $\vec{B}$ using the chain rule. For example

$$\frac{d}{dx}(f g) = \left(\frac{df}{dx}\right) g + f \frac{dg}{dx}$$

The same expression can be symbolically written as

$$\left(\frac{d}{dx}\right) (f g) = \left(\frac{d}{dx}\right)_f (f g) + \left(\frac{d}{dx}\right)_g (f g)$$

where, by definition:

$\left(\frac{d}{dx}\right)_f$ operates only on f

and $\left(\frac{d}{dx}\right)_g$ operates only on g

In the same way

$$\vec{\nabla} \times (\vec{F} \times \vec{G}) = \vec{\nabla}_F \times (\vec{F} \times \vec{G}) + \vec{\nabla}_G \times (\vec{F} \times \vec{G})$$

Now in each of these consider $\vec{\nabla}$ to be a vector:

$$\vec{\nabla}_F \times (\vec{F} \times \vec{G})$$

$$= \vec{F} (\vec{\nabla}_F \cdot \vec{G}) - \vec{G} (\vec{\nabla}_F \cdot \vec{F})$$

$$= (\vec{G} \cdot \nabla) \vec{F} - \vec{G} (\vec{\nabla} \cdot \vec{F})$$

From the second term

$$\vec{\nabla}_G \times (\vec{F} \times \vec{G})$$

$$= \vec{F}(\vec{\nabla}_G \cdot \vec{G}) - \vec{G}(\vec{\nabla}_G \cdot \vec{F})$$

$$= (\vec{\nabla} \cdot \vec{G}) \vec{F} - (\vec{F} \cdot \vec{\nabla}) G$$

Putting together

$$\nabla \times (\vec{F} \times \vec{G})$$

$$= (\vec{G} \cdot \vec{\nabla}) \vec{F} - (\vec{F} \cdot \vec{\nabla}) \vec{G}$$

$$+ (\vec{\nabla} \cdot \vec{G}) \vec{F} - (\vec{\nabla} \cdot \vec{F}) \vec{G}$$