Lecture 5

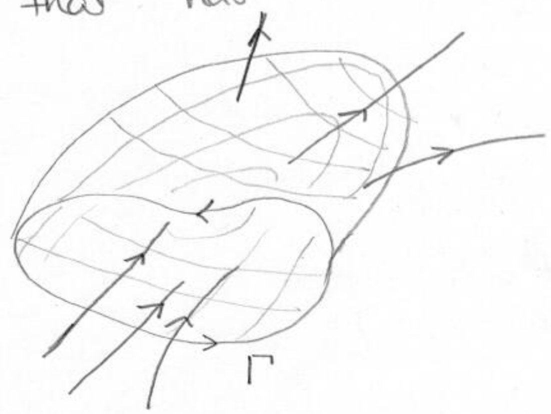
Magnetostatics, further applications

5.1 Magnetic field from current

Ampere's law tells us that the line

$$\oint_{\Gamma} \vec{B} \cdot d\vec{l} = \mu_0 I$$

where Γ is a closed path and I is the total current passing through any suface that has Γ as its perimeter



Using Stokes's theorem,

$$\oint_{\Gamma} \vec{F} \cdot d\vec{l} = \int_{S} \left(\vec{\nabla} \times \vec{F}\right) \cdot \hat{n} \, ds$$

we conclude that,

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$

where $\vec{J}$ is the volume density of current.

This is a local relation where, if we know the dependence of $\vec{B}$ on the space coordinate we can find out the local current density.

The analogy is with the differential form of Gauss's law in electrostatics

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

We also know that,

$$\vec{E}(\vec{R}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho \hat{r}}{r^2} \, dV$$

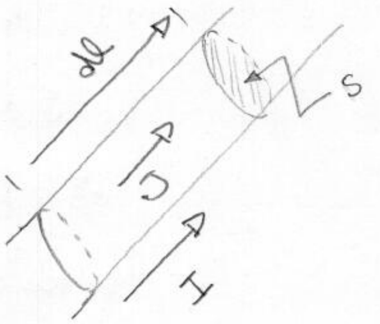
— Coulomb's law.

The second eqn. is clearly the inverse of the first.

What do we get if we invert Ampere's law? We have shown that we obtain the law of Biot and Savart.

$$\vec{B}(\vec{R}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{r}}{r^2} \, dV$$

Sometime, instead of the volume current density we know the current flowing through a circuit. What form does the Biot-Savart law takes then?



The volume element

$$dV = S\, d\ell$$

$$\vec{J}\, dV = J S\, d\vec{\ell}$$

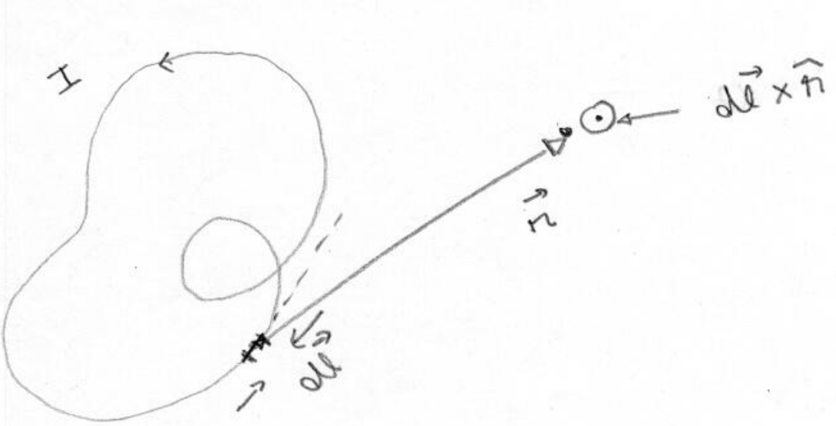
if $\vec{J}$ is essentially constant inside, which is not a bad assumption.

But

$$JS = I$$

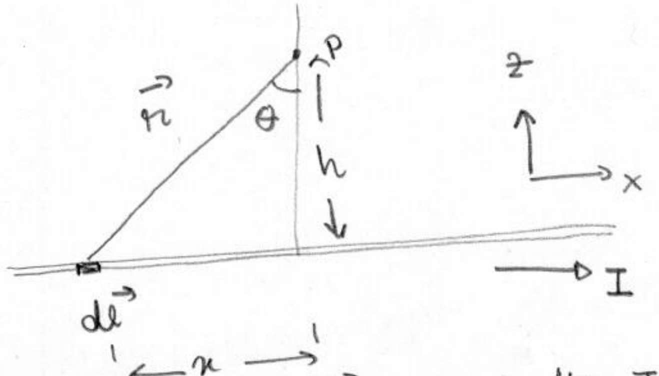
Hence, the Biot-Savart law reads.

$$\vec{B}(\vec{R}) = \frac{\mu_0}{4\pi} \int I \frac{d\vec{\ell} \times \hat{r}}{r^2}$$



Example 5.1

Magnetic field of a line current



$$d\vec{B}(P) = \frac{\mu_0}{4\pi} I \int_{-\infty}^{\infty} \frac{dx \; \hat{x} \times \hat{r}}{r^2}$$

$\hat{x}$
$\hat{z} \circlearrowright \hat{y}$

$$\vec{r} = \hat{x}\, x + \hat{z}\, h$$

$$r^2 = x^2 + h^2$$

$$\hat{r} = \hat{x} \frac{x}{(x^2+h^2)^{1/2}} + \hat{z} \frac{h}{(x^2+h^2)^{1/2}}$$

$$\hat{x} \times \hat{r} = 0 + (-\hat{y}) \frac{h}{(x^2+h^2)^{1/2}}$$

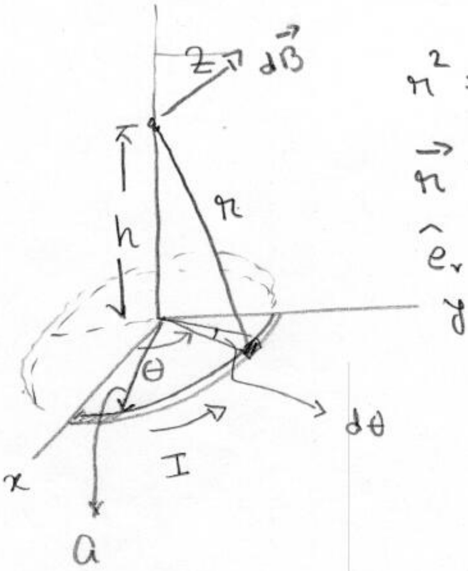
$$\vec{B} = \int d\vec{B} = \frac{\mu_0 I}{4\pi} (-\hat{y})\, h \int_{-\infty}^{+\infty} \frac{dx}{(x^2+h^2)^{3/2}}$$

$$= \frac{\mu_0 I}{4\pi} (-\hat{y})\, h . \frac{2}{h^2} = (-\hat{y}) \frac{\mu_0}{4\pi} \frac{2I}{h}$$

we had obtained the same result before by using Ampere's law and symmetry.

Example 5.2

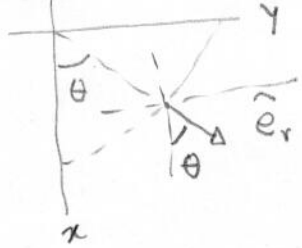
Magnetic field of a current loop on the axis



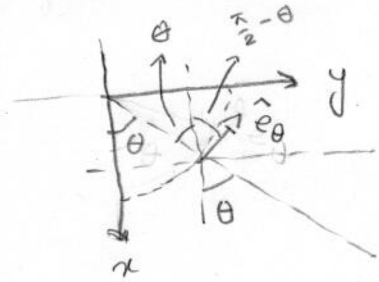
$$r^2 = a^2 + h^2$$

$$\vec{r} = -\hat{e}_r a + \hat{z} h$$

$$\hat{e}_r = \hat{x} \cos\theta + \hat{y} \sin\theta$$



Let us use the cylindrical coordinate first;



$$d\vec{l} = a\,d\theta\,\hat{e}_\theta$$

$$\hat{e}_\theta = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$d\vec{B} = \frac{\mu_0 I a}{4\pi} \int_0^{\pi} \frac{\hat{e}_\theta \times (-\hat{e}_r a + \hat{z} h)\, d\theta}{(a^2 + h^2)^{3/2}}$$

$$\hat{e}_\theta \times (-a\hat{e}_r + h\hat{z}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\sin\theta & \cos\theta & 0 \\ -a\cos\theta & -a\sin\theta & h \end{vmatrix}$$

$$= \hat{x}(-a\sin\theta) + \hat{y}(h\sin\theta) + \hat{z}(a\sin^2\theta + a\cos^2\theta)$$

$$= \hat{x}(-a\sin\theta) + \hat{y}(h\sin\theta) + a\hat{z}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \frac{a}{\left(a^2+h^2\right)^{3/2}} \left[-a\hat{x} \int_0^{2\pi} \sin\theta \, d\theta \right.$$

$$+ h\hat{y} \int_0^{2\pi} \cos\theta \, d\theta$$

$$\left. + a\hat{z} \int_0^{2\pi} d\theta \right]$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \frac{a^2 2\pi \, \hat{z}}{\left(a^2+h^2\right)^{3/2}}$$

For $h = 0$ (at the center of the loop)

$$\vec{B} = \frac{\mu_0 I}{4\pi} \frac{2\pi}{a} \hat{z}$$

For large h,

$$\frac{1}{\left(a^2+h^2\right)^{3/2}} = \frac{1}{h^3} \frac{1}{\left(1 + a^2/h^2\right)^{3/2}} = \frac{1}{h^3}\left(1+\xi^2\right)^{-3/2}$$

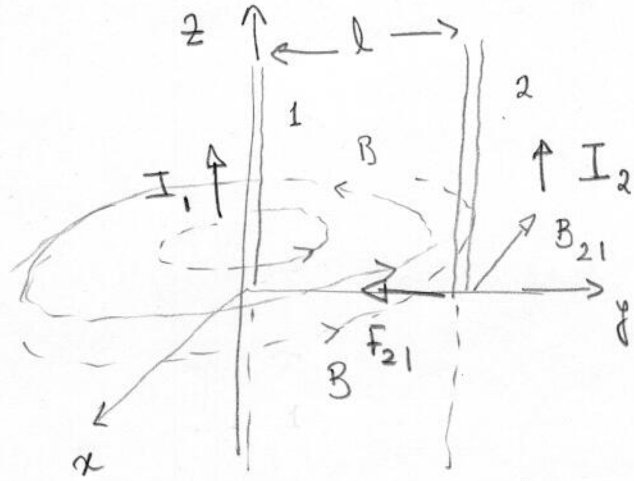
$$\simeq \frac{1}{h^3}\left(1 - \frac{3}{2}\xi^2 + \cdots\right)$$

$$= \frac{1}{h^3}\left(1 - \frac{3}{2}\frac{a^2}{h^2} + \cdots\right)$$

$$\vec{B} \simeq \frac{\mu_0}{4\pi} \frac{\pi a^2 I}{h^3}\hat{z} \doteq \frac{\mu_0}{4\pi}\frac{m}{h^3}\hat{z} \qquad \underbrace{m = I a^2 \pi}_{\text{magnetic dipole moment.}}$$

5·2

Force between two current carrying wires.



By Newton's third law,

$$\begin{pmatrix} \text{force on 1} \\ \text{due to 2} \end{pmatrix} = \begin{pmatrix} \text{force on 2} \\ \text{due to 1} \end{pmatrix}$$

$$\begin{pmatrix} \text{B at 2} \\ \text{due to 1} \end{pmatrix} = \begin{pmatrix} \text{B of a straight} \\ \text{wire at a distance } l \\ \text{from the wire} \end{pmatrix}$$

$$\vec{B}_{21} = \frac{\mu_0}{4\pi} \frac{2I_1}{l}\left(-\hat{x}\right) \quad \text{[Example 5·1]}$$

$$\begin{pmatrix} \text{force on 2} \\ \text{due to 1} \end{pmatrix} = \begin{pmatrix} \text{force on a current carrying} \\ \text{wire in field B at 2} \end{pmatrix}$$

$$= -I_2 B_{21}\,\hat{y} \quad \text{(per unit length)}$$

[section 4·3]

$$F_{21} = \frac{\mu_0}{4\pi} \frac{2I_1 I_2}{l} \quad \text{(per unit length of 2)}$$

Comments

- The force is attractive if both the currents flow in the same direction.

- How large is this force?

Let two wires of current 1A each is placed 1 cm apart, then

F (per unit length)

$$= 10^{-7} \frac{2.}{10^{-2}} \text{ Newton } m^{-1}$$

$$= 2 \times 10^{-5} \text{ Newton. } m^{-1}$$

consider a copper wire of length 1 m and the cross-section is a circle of radius 0.5 mm. The weight of the wire

$$F_g = \pi\left(\frac{1}{2} \times 10^{-3}\right)^2 (1) \underbrace{\frac{8.96 \times 10^{-3}}{10^{-6}} \frac{kg}{m^3}}_{\text{density of copper}} \times \overbrace{(10)}^{g} \text{ Newton.}$$

$$F_g = \pi \frac{1}{4} \, 9 \times 10 \times 10^{-3} \text{ Newton}$$

$$\simeq 0 \cdot 1 \quad \text{Newton}$$

The magnetic force is about 10^{-4} times the weight but still measurable.

Consider the fact that in one case we consider the gravity due to the earth, whereas in the magnetic case we consider a meter-long wire. From this angle the magnetic force is huge compared to the gravitational force.

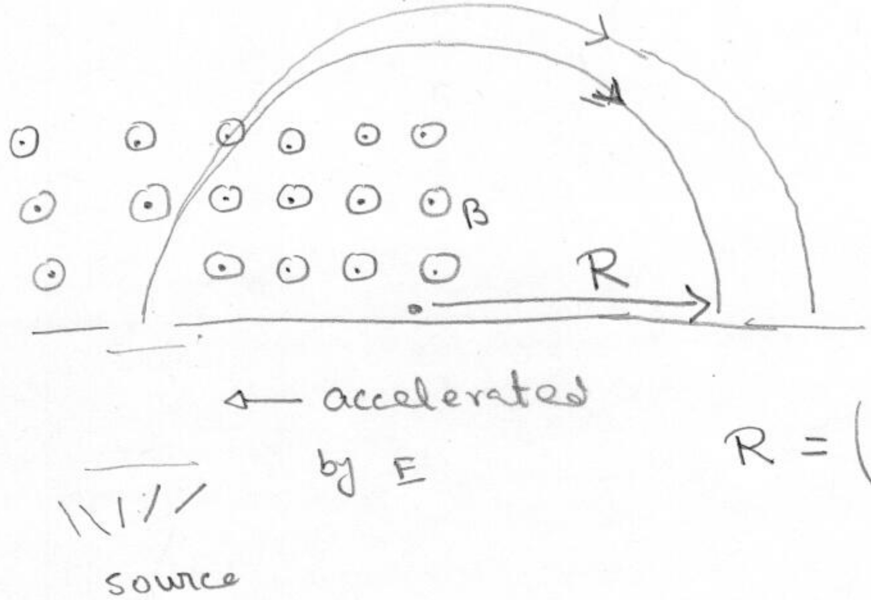
- The eventual force is calculated using two cross-products: $\vec{dl} \times \hat{r}$ in the Biot-Savart law and the $\vec{v} \times \vec{B}$ in the Lorentz force law. If instead of the right-handed rule we used the left handed rule in both cases we would get the same attractive force. Hence nature does not distinguish between the left and the right hand; the rules are mere conventions. (At least in classical EM)

5.3 Isotope separation. (mass spectrometer)

The gyroradius

$$R = \frac{mv}{qB}$$

Isotopes are atoms that have the same number of protons (same atomic number) but different number of neutrons. Two isotopes have exactly the same chemical property (because they have the same number of orbital electron) but may have very different nuclear property, one may be radioactive the other may not. For applications in nuclear reactors we may need to enrich one isotope compared to the other. One way to separate them is to send them through the same magnetic field



$$R = \left(\frac{m}{q}\right)\left(\frac{v}{B}\right)$$

The lighter isotope has smaller R.

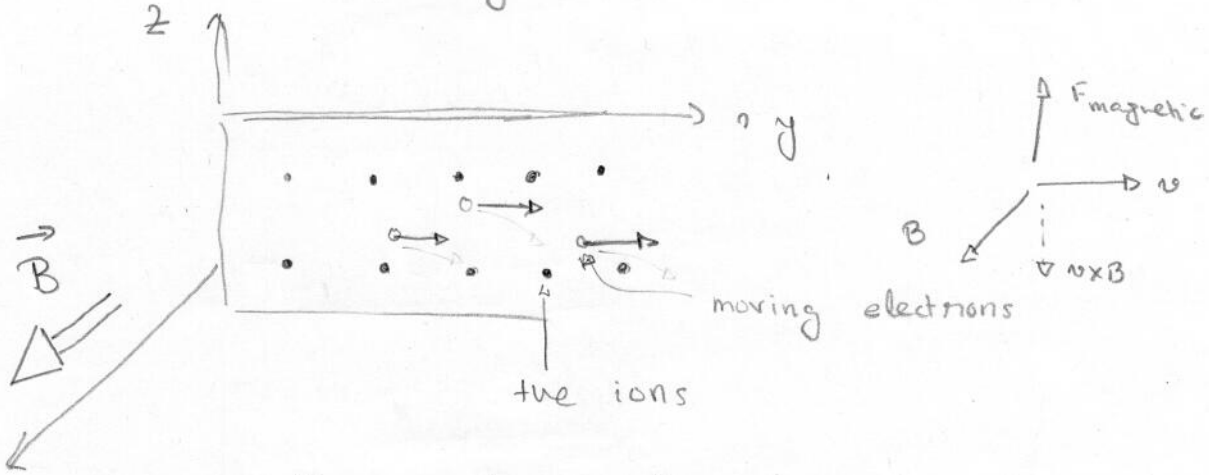
Consider the isotopes U^{235} and U^{238}.

U^{235} will be collected at the "near" point.

This method of isotope separation is not industrially viable because the througput is small but was used to enrich U^{235} for the first atomic bomb.

5.4 Hall effect

We know that a current carrying wire experiences a force in the presence of a magnetic field. Let us look at the problem microscopically.



Because the magnetic force act on the electrons they get force $\vec{v} \times \vec{B}$ in the -ve z direction. Then a current flows in the z direction. This sets up negative surface charges at the top surface of the metallic conductor. This process in a transient and happens quite fast. The surface charges pile up till the electric field due to the surface charges is equal to the magnetic force. After which the electrons are back on the same track* that they moved before the magnetic field was turned on. The electric field due to the surface charges is such that:

$$\vec{E}_s + \vec{v} \times \vec{B} = 0$$

Clearly the current

$$\vec{J} = nq\vec{v}$$

$$\Rightarrow \boxed{\vec{E}_s + \frac{\vec{J} \times \vec{B}}{nq} = 0}$$

The potential difference due to the electric field can be measured by a voltmeter. This effect can be used to find out the sign of the moving charges in a metal. Furthermore can be used to measure magnetic fields, once calibrated.

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* One cannot actually assign a track to an electron but that issue can be ignored here.