Magnetostatics

4.1 Early experiments with permanent magnets and currents in wires showed that magnets can exert force on a current carrying wire. Without going into all the details of experiments let us write down the law :

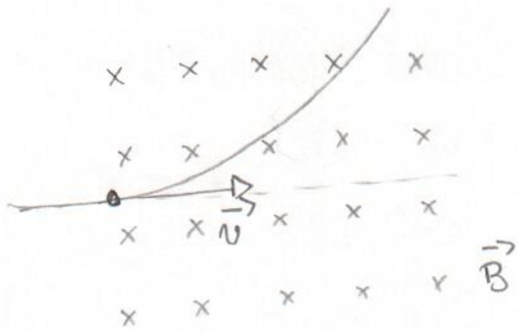
The force on a charged particle of charge q moving with a velocity $\vec{v}$ in the presence of a magnetic field $\vec{B}$ is given by

$$\boxed{\vec{F} = q\, \vec{v} \times \vec{B}}$$

In the presence of an electric field $\vec{E}$ the force is

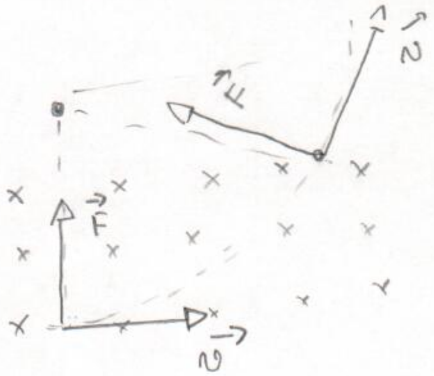
$$\boxed{\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)}$$

— Lorentz force



Example 4·1

Trajectory of a charged particle in a constant magnetic field.



Use $\vec{B} = B_0 \hat{z}$

Let the velocity be in $x-y$ plane:

$$\vec{v} = \hat{x} v_x + \hat{y} v_y$$

Then $\vec{F} = q(\vec{v} \times \vec{B}) = q \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & 0 \\ 0 & 0 & B_0 \end{vmatrix}$

$$= q\left(\hat{x} B_0 v_y - \hat{y} B_0 v_x\right)$$

$$= q B_0 \left(\hat{x} v_y - \hat{y} v_x\right)$$

$$\vec{F} = m\left(\hat{x} \frac{d^2 x}{dt^2} + \hat{y} \frac{d^2 y}{dt^2} + \hat{z} \frac{d^2 z}{dt^2}\right)$$

$$\frac{d^2z}{dt^2} = 0 \, , \quad \Rightarrow \quad z = 0$$

$$\frac{d^2x}{dt^2} = \frac{qB_0}{m} v_y \quad , \quad \frac{d^2x}{dt^2} = \dot{v}_x \, , \quad \frac{d^2y}{dt^2} = \dot{v}_y$$

$$\frac{d^2y}{dt^2} = -\frac{qB_0}{m} v_x$$

$$\Rightarrow \quad \dot{v}_x = \frac{qB_0}{m} v_y \, , \qquad \dot{v}_x = \omega v_y$$

$$\dot{v}_y = -\frac{qB_0}{m} v_x \, , \qquad \dot{v}_y = -\omega v_x$$

dimensionally $\quad \dfrac{qB_0}{m} = \dfrac{1}{\text{time}} = \text{cyclotron frequency} = \omega$.

One clever way of dealing with this problem is to substitute

$$\dot{v}_x + i\,\dot{v}_y = \omega v_y - i\omega v_x$$

$$= -i\omega\left(v_x + i v_y \right)$$

$$G = v_x + i v_y$$

$$\frac{dG}{dt} = -i\omega G \Rightarrow G(t) = G(0)\, e^{-i\omega t}$$

$$= G(0)\left[\cos\omega t - i \sin\omega t\right.$$

$$\left.\begin{aligned} v_x &= v_x(0)\cos\omega t \\ v_y &= -v_y(0)\sin\omega t \end{aligned}\right\}$$

$\dot{x} = v_x(0)\cos\omega t$ $\qquad$ $\dot{y} = -v_y(0)\sin\omega t$

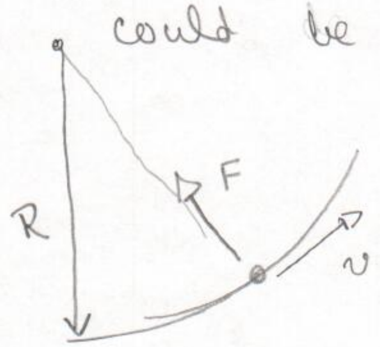
$x = v_x(0)\dfrac{\sin\omega t}{\omega}$ $\qquad$ $y = -v_y(0)\dfrac{\cos\omega t}{\omega}$

$$x^2 + y^2 = \frac{v^2}{\omega^2}$$

– eqn. of a circle

$$\boxed{R = \frac{v}{\omega} = \frac{vm}{qB_0}}$$ – gyro radius.

could be derived much more simply



centripetal acceleration $\dfrac{v^2}{R}$

$$\frac{mv^2}{R} = qvB \Rightarrow \boxed{R = \frac{mv}{qB}}$$

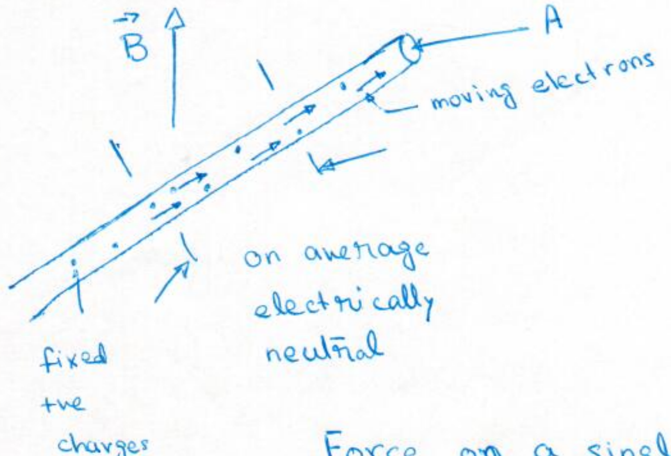
4.2 Magnetic forces do no work.

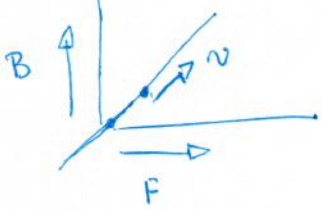
$$\vec{F} = q\left(\vec{v} \times \vec{B}\right)$$

Rate of work done

$$\vec{F} \cdot \vec{v} = q\,\vec{v} \cdot \left(\vec{v} \times \vec{B}\right) = 0$$

4.3 Magnetic force on a current carrying wire.





Force on a single charge q

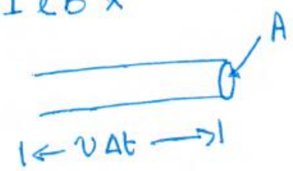
$$\vec{F}_i = q\, \vec{v} \times \vec{B}$$

In a wire of length l and area A

The number of charges per unit volume n

The force on a wire of length l.

$$\vec{F} = \sum F_i = n q\, \vec{v} \times \vec{B}\, l A = I l B \hat{x}$$



The current $I = A n q \dfrac{v \Delta t}{\Delta t}$

$$\underset{\text{Newton}}{\vec{F}} = \underset{\text{coulomb}}{q}\, \overset{m s^{-1}}{\vec{v}} \times \underset{\text{Tesla}}{\vec{B}}$$

1 Tesla $= 10^4$ Gauss

Magnetic field of earth on its surface is few tenth of Gauss.

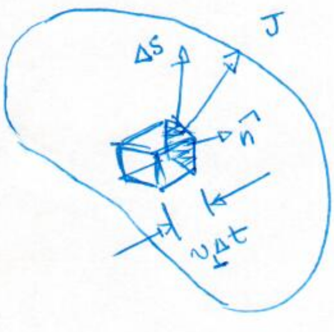
Magnetic field in a superconducting magnet (like a MRI machine) few tesla.

Magnetic field in a sunspot $\sim 10^3$ Gauss

magnetic field on the surface of a Neutron star $\sim 10^{12}$ gauss.

Galactic magnetic field $\sim$ ' μgauss.

4·4 Current density



The flux in time Δt

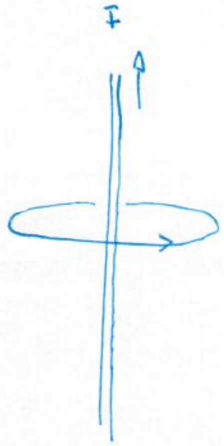
$$\rho v_\perp \Delta t \Delta S = \vec{J} \cdot \hat{n} \Delta S$$

$$\boxed{\vec{J} = \rho \vec{v}}$$

4.5

Ampere's law:

Magnetic field due to current carrying wire



$$\oint_{\Gamma} \vec{B} \cdot \vec{dl} = \mu_0 I_{enc}$$

μ_0 $\downarrow$ permeability of vacuum.

$$\oint \vec{B} \cdot \vec{dl} = 0$$

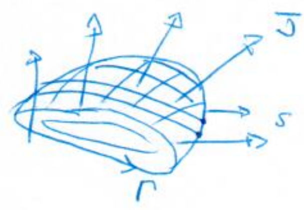
Any loop however shaped gives the same result.

The analogy is with Gauss's law

$$\oint_S \vec{E} \cdot \hat{n} \, ds = \frac{\Phi_{enc}}{\epsilon_0}$$

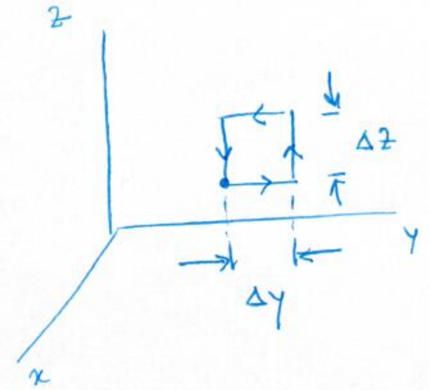
Generalized to a continuum charge distribution

$$\oint_{\Gamma} \vec{B} \cdot \vec{dl} = \int_S \mu_0 \vec{J} \cdot \hat{n} \, ds$$



$$\boxed{\mu_0 = 4\pi \times 10^{-7} \text{ SI}}$$

4.6 Line integral of a vector field

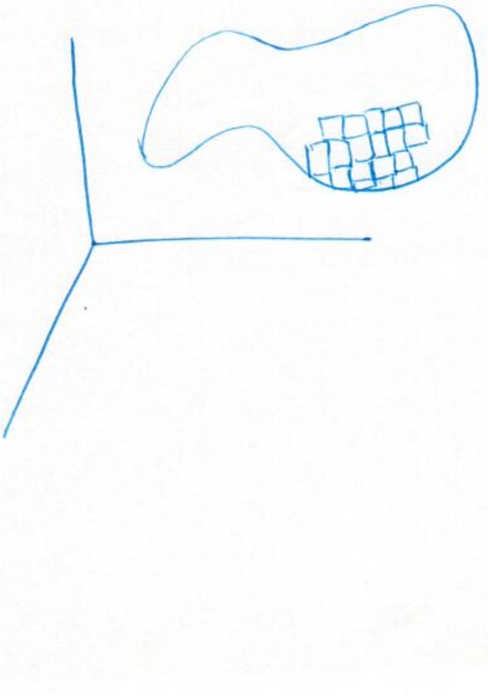


$$\oint \vec{F}\cdot dl = F_y(x,y,z)\,\Delta y - F_y(x,y,z+\Delta z)\,\Delta y - F_z(x,y,z)\,\Delta z + F_z(x,y+\Delta y,z)\,\Delta z$$

$$= \left[F_y(x,y,z) - F_y(x,y,z) \mp \frac{\partial F_y}{\partial z}\Delta z\right]\Delta y$$

$$+ \left[F_z(x,y,z) - F_z(x,y,z) + \frac{\partial F_z}{\partial y}\Delta y\right]\Delta z$$

$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\Delta y\,\Delta z$$



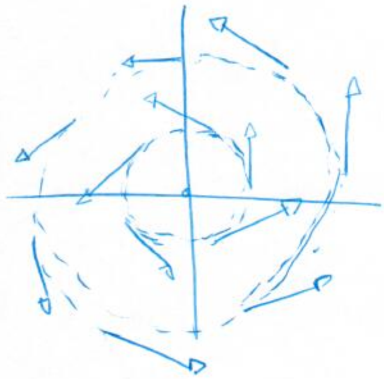
$$\oint_\Gamma \vec{F}\cdot d\vec{l} = \int_S \left(\vec{\nabla}\times\vec{F}\right)\cdot\hat{n}\,ds$$

— Stokes theorem.

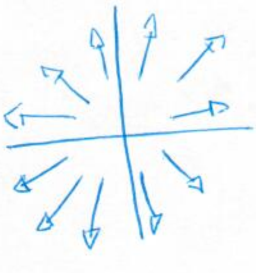
4.7 curl of a vector field

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \hat{x}\left(\partial_y F_z - \partial_z F_y\right)$$
$$+ \hat{y}\left(\partial_z F_x - \partial_x F_z\right)$$
$$+ \hat{z}\left(\partial_x F_y - \partial_y F_x\right)$$



curl.



divergence

4.8 vector differential operators

$$\vec{\nabla} = \left(\hat{i}\, \partial_x + \hat{j}\, \partial_y + \hat{k}\, \partial_z \right)$$

- For any scalar function ψ

$$\nabla \times \nabla \psi = 0$$

proof:

Use cartesian coordinates

$$\nabla \times \nabla \psi = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ \partial_x \psi & \partial_y \psi & \partial_z \psi \end{vmatrix}$$

$$= \hat{x} \left(\partial_y \partial_z \psi - \partial_z \partial_y \psi \right) + \quad +$$

$$= 0.$$

$\Rightarrow$ If for any vector function $\vec{F}$,

$$\vec{\nabla} \times \vec{F} = 0$$

$$\Rightarrow \quad \vec{F} = \vec{\nabla} \psi$$

An example is the electrostatic field

$$\oint_{\Gamma} \vec{E} \cdot d\vec{l} = 0$$

$$\Rightarrow \quad \nabla \times \vec{E} = 0 , \quad \Rightarrow \quad \boxed{\vec{E} = - \vec{\nabla} \phi}$$

A far simpler proof

$$\oint (\nabla \psi) \cdot d\vec{l} = 0$$

$$\Rightarrow \quad \nabla \times \nabla \psi = 0$$

4.9

For any vector field $\vec{F}$

$$\vec{\nabla} \cdot \bar{\nabla} \times \bar{F} = 0$$

proof Use cartesian coordinates and proceed in a straightforward manner.

Γ

$$\oint \vec{F} \cdot \vec{dl} = \int_S (\bar{\nabla} \times \bar{F}) \cdot \hat{n}\, ds$$

As you go to a closed surface:

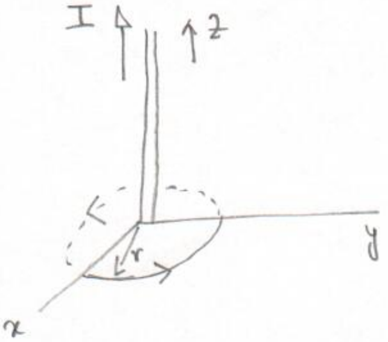
$$\oint_S (\bar{\nabla} \times \bar{F}) \cdot \hat{n}\, ds = \int_V \operatorname{div} (\bar{\nabla} \times \vec{F})\, dv = 0$$

As the ~~loop~~ surface becomes closed the loop Γ shrinks to point and the path integral goes to zero.

4.10

Applications of Ampere's law

- Magnetic field of a long straight wire:

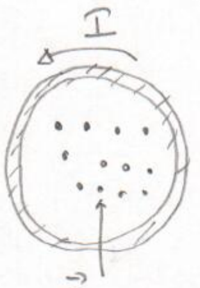


$$\oint_{\Gamma} \vec{B}\cdot d\vec{l} = 2\pi r B_\phi$$

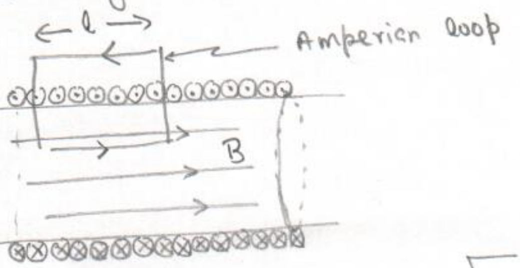
$$\Rightarrow \quad 2\pi r B_\phi = \mu_0 I$$

$$\Rightarrow \quad B_\phi = \frac{\mu_0 I}{2\pi r}$$

- Magnetic field of a long solenoid



$\vec{B}$ is out of the plane

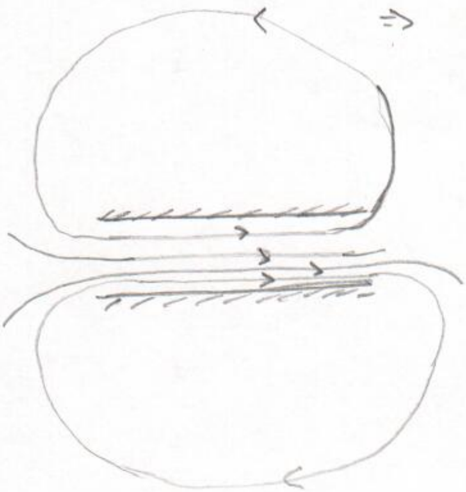


$$\oint \vec{B}\cdot d\vec{l} = Bl = \mu_0 N I l$$

turns per unit length (N)

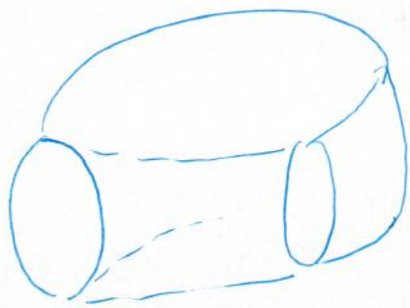
current (I)

$$\Rightarrow \quad \boxed{B = \mu_0 N I}$$



Field of a real solenoid.

- Magnetic field of a torroid.



A torroid is a solenoid closed onto itself. Take an Ampearian loop along its axis (which is a circle) to obtain the same result as an infinitely long solenoid.

4.11 There are no magnetic monopoles.

$$\oint_S \vec{B}\cdot\hat{n}\,ds = 0 \iff \operatorname{div}\vec{B} = 0$$

Consequence

$$\vec{B} = -\vec{\nabla}\times\vec{A}$$

As $\nabla\times\vec{B} = \mu_0\vec{J}$

$$-\nabla\times\nabla\times\vec{A} = \mu_0\vec{J}$$

$$\Rightarrow \quad -\nabla(\nabla\cdot\vec{A}) + \nabla^2\vec{A} = \mu_0\vec{J}$$

choose $\vec{A}$ such that $\nabla\cdot\vec{A} = 0$

$$\Rightarrow \quad \boxed{\nabla^2\vec{A} = \mu_0\vec{J}}$$

~~vec~~ vector identity

$$\nabla \times \nabla \times \vec{A} = \vec{\nabla}(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

proof

1. Use cartesian coordinates, expand and collect terms and show the identity.

2. $\vec{\nabla}$ is a vector.

For any three vectors we know the identity:

$$\vec{A} \times (\vec{B} \times c) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

put $\vec{C} = \vec{F}$,

$\vec{B} = \nabla$

$\vec{A} = \nabla$

and remember that ∇ is also a differential operator so it must act on something on its right.

$$\Rightarrow \quad \nabla \times \nabla \times \vec{F} = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{F}$$

$$= \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F}$$

4.12 The magnetic vector potential.

$$\nabla^2 \vec{A} = \mu_0 \vec{J}$$

This implies three scalar equations

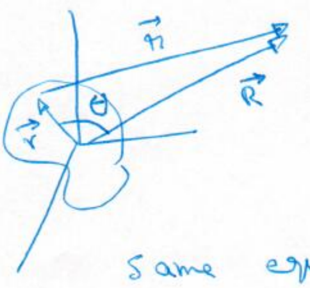
$$\nabla^2 A_x = \mu_0 J_x, \quad \nabla^2 A_y = \mu_0 J_y, \quad \nabla^2 A_z = \mu_0 J_z$$

we have already seen one such equation before

$$\nabla^2 \phi = \frac{\rho}{\epsilon_0}$$

which has the solution

$$\phi(\vec{R}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r})\, dV}{\eta} \qquad \vec{R} = \vec{r} + \vec{\eta}$$



same equations always have same solution

$$\Rightarrow \quad \boxed{\vec{A}(\vec{R}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}\, dV}{\eta}}$$

and

$$\vec{B}(\vec{R}) = \frac{\mu_0}{4\pi} \underbrace{\nabla\times}_{\text{target}} \int \underbrace{\frac{\vec{J}(\vec{r})\, dV}{\eta}}_{\text{source}}$$

$$= \frac{\mu_0}{4\pi} \int J(\vec{r})\, dV \;\; \nabla\times\left(\frac{1}{\eta}\right)$$

$$= \frac{\mu_0}{4\pi} \int J(\vec{r})\, dV \;\; \nabla\times\left(\frac{1}{r^2 + R^2 - 2Rr\cos\theta}\right)$$

consider $\nabla \times (\phi \vec{A})$ where $\vec{A}$ is a constant vector

By the chain rule

$$\nabla \times (\phi \bar{A}) = (\nabla \phi) \times \bar{A} + \phi (\nabla \times A)$$

$$= (\nabla \phi) \times \vec{A}$$

Remember ∇ is both a vector and a differential operator.

$$\nabla \times \frac{\vec{J}(\vec{r})}{r}$$

$$= \vec{\nabla}\left(\frac{1}{r}\right) \times \vec{J}$$

$$= -\frac{1}{r^2}\, \hat{r} \times \bar{J} = \frac{\bar{J} \times \hat{r}}{r^2}$$

$$\Rightarrow \quad \boxed{\vec{B}(\vec{R}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J} \times \hat{r}}{r^2}\, dV}$$

– Law of Biot and Savart.

comments

- vector potential and choice of gauge.

$$\nabla \cdot A = 0$$

- Is vector potential real or just a mathematical construction?

- Are fields real?