Electrostatics

Lecture 3.

3.1 Electrostatics with conductors

In the early days of experiments with electricity people knew that certain materials when charged can hold their charge for a significant amount of time. Whereas other materials will loose their charge quite fast. The example of the first is a glass sphere. The second is metal. Now we call them insulators and conductors. If you want to perform an experiment with a charged metal sphere you need to isolate it with a glass holder.

The difference between a conductor and an insulator is one of the most dramatic differences found in nature. It is similar to the flow properties of solid and liquid. In nature there also exists substances with intermediate conductivity properties. Also conductivity can depend crucially on temperature.

For the rest of this lecture we shall concentrate on electrostatics of conductors. Only one property of conductors need to concern us here. The conductors have "free charges" inside them such that when placed in an electric field the free charges moves to create an induced charge distribution such that the electric field inside the "meat" of the conductor is zero. This is the only way we can have electrostatics in conductors because if $\vec{E}$ was not zero the free charges would move and it would not be <u>statics</u> anymore.

- Field inside a conductor is zero.

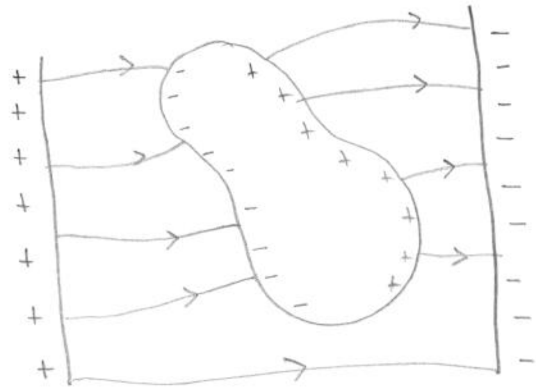
<u>consequence</u>

(a) The charge density inside a conductor is also zero. Because,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

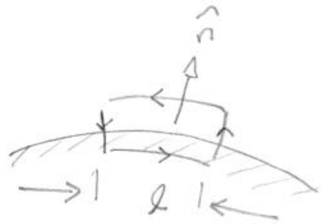
$\vec{E}$ is everywhere zero inside a conductor so $\vec{\nabla} \cdot \vec{E} = 0$ everywhere inside the conductor too.

(b) Any charges must reside on the surface of the conductor.



(c) The electric near the surface of the conductor is perpendicular to its surface and is given by $E_n = \frac{\sigma}{\epsilon_0}$

proof



$$\oint \vec{E} \cdot \vec{dl} = 0$$

$E_t \, l = 0$ for any l

$$\Rightarrow \boxed{E_t = 0}$$

$$\oint_S \vec{E} \cdot \hat{n} \, ds = \frac{\sigma}{\epsilon_0}(\delta s)$$

$$E_n \, \delta S = \frac{\sigma}{\epsilon_0} \delta S$$

$$\Rightarrow \boxed{E_n = \frac{\sigma}{\epsilon_0}}$$

(d) The surface of a conductor is an equipotential. Otherwise there would be electric field parallel to the surface.

(e) The surface charge density of a conductor is inversely proportional to the local radius of curvature.

proof consider a sphere charged to potential V. If the total charge on the sphere is q then:

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{a}$$

The surface electric field

$$\vec{E} = \hat{r}\,\frac{1}{4\pi\epsilon_0}\frac{q}{a^2}$$

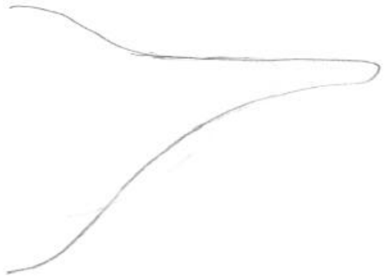
The surface charge density

$$\sigma = \epsilon_0 E_n = \frac{q}{4\pi a^2}$$

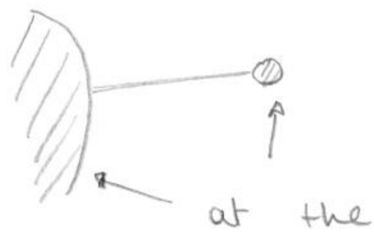
$$= \frac{\epsilon_0 V a}{a^2} \Rightarrow \boxed{\sigma = \frac{\epsilon_0 V}{a}}$$

smaller the sphere higher the surface charge density

Now consider a conductor shaped like



This can be approximated by



Hence the tip is going to have a very high charge density. And consequently a very high electric field. At the tip of a needle the field can be so high that air can "break down"

If a small amount of He is introduced in the chamber the He atoms can, by random motion, hit the tip and get ionized. Then the He ion will be moved by the strong electric field and hit the surface of the bulb. This way the atomic level structure of the tip can be mapped out.

(f) Field inside the cavity inside a conductor is zero due to external charges.

$$\oint \vec{E} \cdot d\vec{l} = 0$$

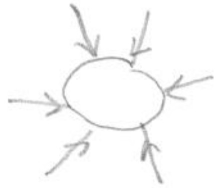
for any path

$$\Rightarrow \quad \vec{E} = 0$$

This is the principle of <u>electrostatic shielding</u>.

3·2 There can be no local equilibrium in electrostatics.

proof If a point is local equi equilibrium then



$$\oint_S \vec{E}\cdot\hat{n}\,ds < 0$$

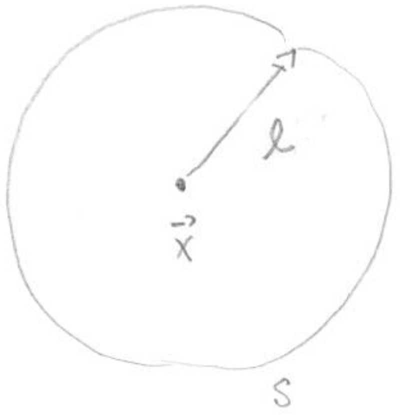
but $\oint_S \vec{E}\cdot\hat{n}\,ds$ must be zero as there are no charges at that point

This also implies that the potential cannot have any local absolute minima of maxima in charge free space.

In other words the solution of Laplace's eqn can have maxima or minima only at the boundaries.

In free space potential satisfies the Laplace's eqn.

$$\nabla^2\phi = 0$$



Average of ϕ over a small spherical surface S will be

$$\langle \phi \rangle = \oint_S \phi(\vec{x}+\vec{r})\, ds$$

$$= \phi(\vec{x}) + \oint (\nabla \phi) \cdot \vec{r}\; ds$$

$$= \phi(\vec{x}) + \oint \vec{E} \cdot \vec{r}\, ds$$

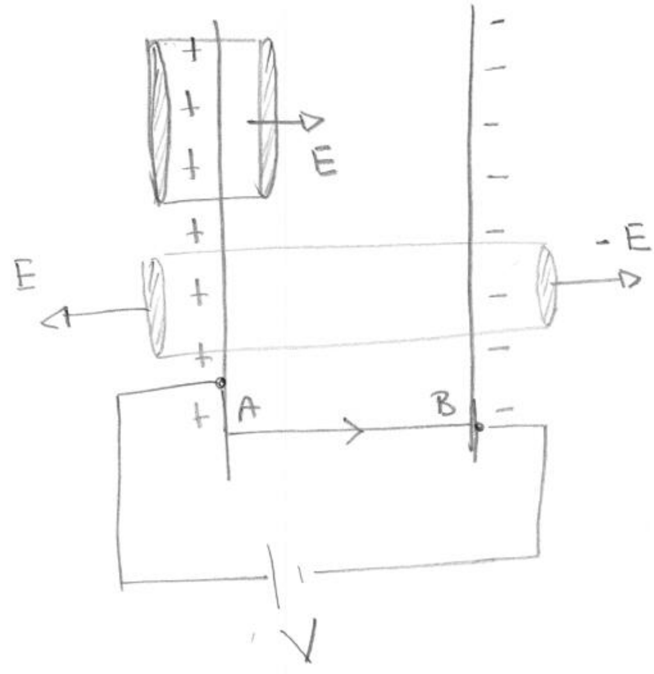
$$= \phi(\vec{x}) + \underbrace{\oint \vec{E} \cdot \hat{n}\, ds}_{=0}$$

The average of ϕ over a small sphere is the same as value of ϕ at the center.

$\Rightarrow$ ϕ cannot have local maxima or minima.

A very useful theorem to find ϕ in complicated geometry by the method of relaxation.

3.3 Capacitors



$$E = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0}$$

$$\left.\begin{aligned} &= \frac{\sigma}{\epsilon_0} \\ &= 0 \end{aligned}\right\} \begin{aligned} &\text{inside} \\ &\text{outside} \end{aligned}$$

$$V = -\int_A^B \vec{E} \cdot d\vec{l}$$

$$\Delta V = \frac{\sigma d}{\epsilon_0}$$

$$\cong (\sigma A)\left(\frac{d}{\epsilon_0 A}\right)$$

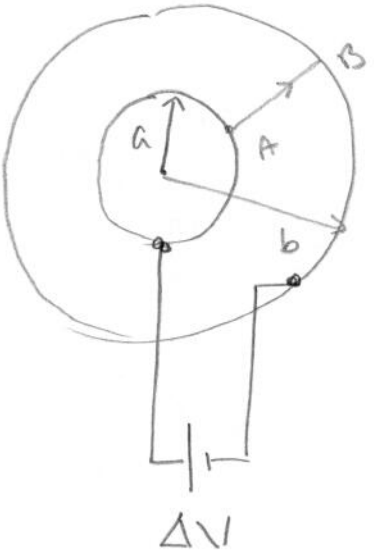
$$\cong Q\left(\frac{d}{\epsilon_0 A}\right)$$

$$\boxed{C = \frac{Q}{\Delta V} = \frac{\epsilon_0 A}{d}}$$

depends only on the geometry of the configuration.

Example 3.1

capacitance of a system of two spherical shells of radius a, and b



$$\Delta V = \int_A^B \vec{E} \cdot d\vec{l}$$

$$= \frac{1}{4\pi\epsilon_0} \int_a^b \frac{Q_{in}}{r^2}\, dr$$

$$\Delta V = \frac{Q_{in}}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$C = \frac{Q_{in}}{\Delta V} = 4\pi\epsilon_0 \frac{1}{(1/a - 1/b)}$$

$$\boxed{C = \frac{4\pi\epsilon_0\, ab}{b-a}}$$

leaving ϵ_0 out c is like a length.

A length of cm gives capacitance of the order of picofarads.

For charge in coulomb, and potential differences in volts capacitance is in farads.

3.4

Energy stored in a capacitor.

Work done to tranport charge dq across potential difference V is $dW = V\,dq$

For a capacitor of capacitance C

$$C = \frac{q}{V} \Rightarrow dq = C\,dV$$

$$\Rightarrow \quad dW = C\,V\,dV$$

Total energy stored

$$U = \int_0^V C\,V\,dV = \frac{1}{2} C V^2$$

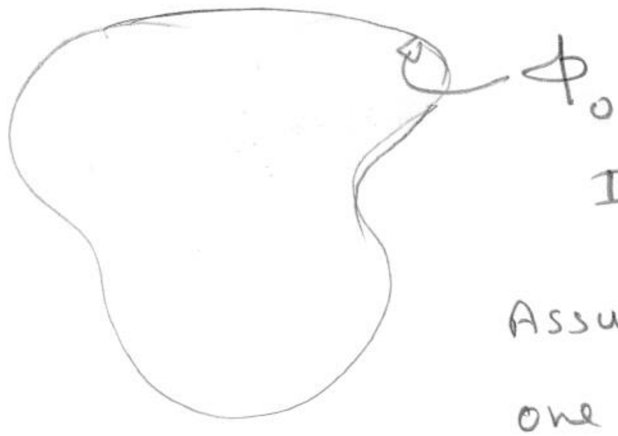
True for any general capacitor system.

3.5

Uniqueness of solution of Laplace's eqn.

If you have found one solution of an electrostatics problem that satisfies the given boundary condition then it must be <u>the</u> only solution.

Assume that ϕ is specified at the boundaries of a volume V.



Inside $\nabla^2\phi = 0$

Assume $\phi = \phi_1$ is one solution and $\phi = \phi_2$ another

They both satisfy the same boundary condition. Then

$$\psi = \phi_1 - \phi_2$$

satisfies the following problem.

$$\nabla^2 \psi = 0$$

$\psi = 0$ on the boundary.

$\Rightarrow$ ψ must be zero everywhere as ψ cannot have any local maxima or minima.

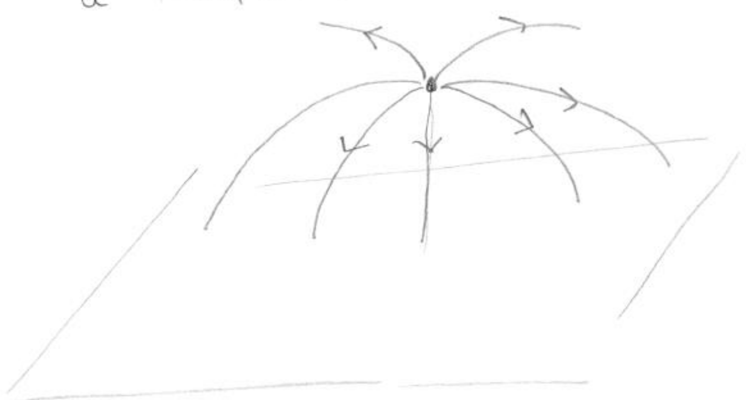
$$\Rightarrow \qquad \phi_1 = \phi_2$$

The solution is unique.

It can be generalized to the case where $\partial_n \phi$ is given on the boundary instead of ϕ itself.

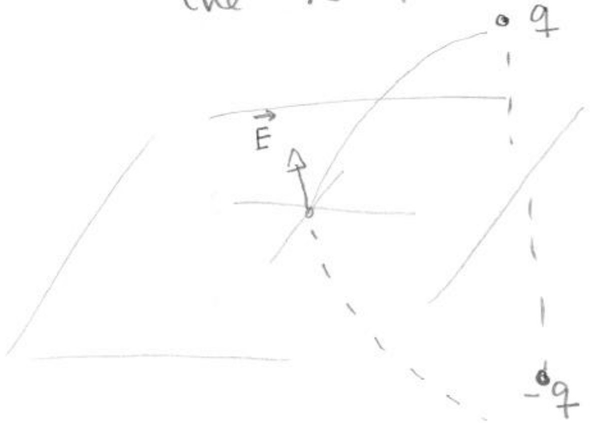
3·6

Field of a point charge before a infinite metal plate.

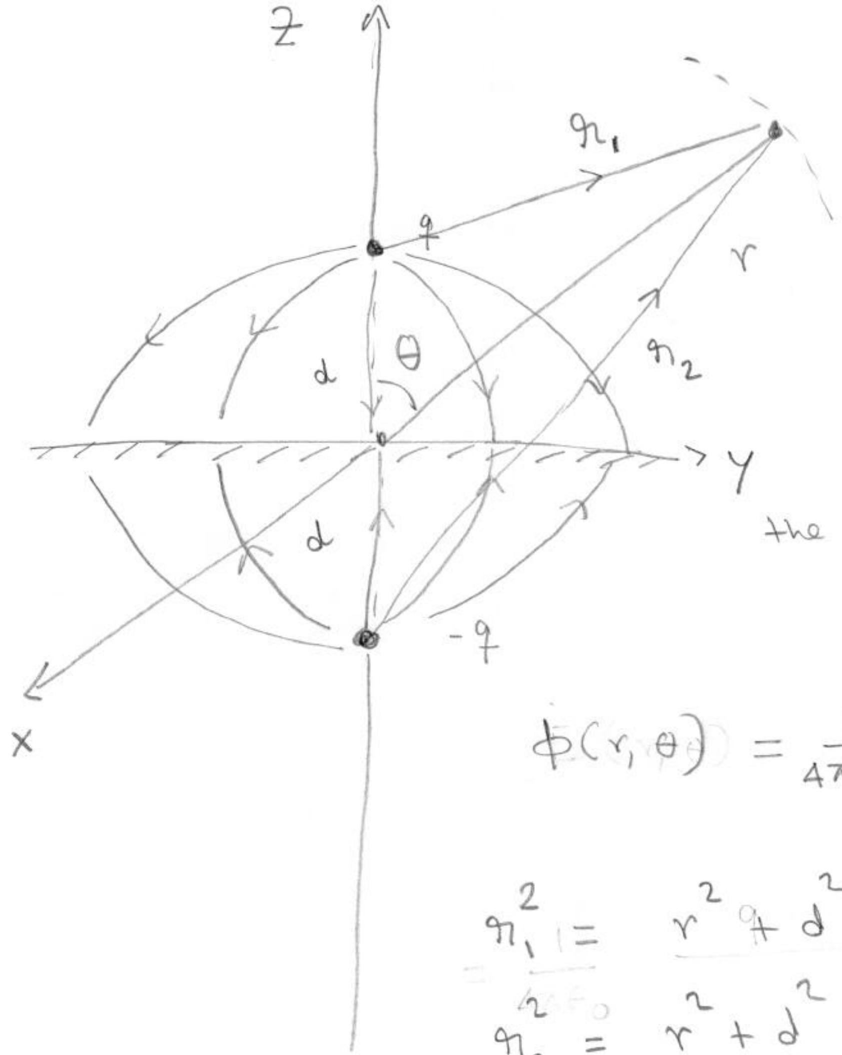


If I can find a solution that satisfies the boundary conditions then that must be the only solution.

The field must be perpendicular to the surface of the plane.



This can be done by adding a charge of $-q$ to the mirror image of the charge.



The field everywhere is the potential of the pair of charges.

$$\Phi(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} - \frac{q}{r_2}\right)$$

$$r_1^2 = r^2 + d^2 - 2dr\cos\theta$$

$$r_2^2 = r^2 + d^2 + 2dr\cos\theta$$

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(r^2+d^2-2dr\cos\theta)^{1/2}} - \frac{1}{(r^2+d^2+2dr\cos\theta)^{1/2}}\right]$$

To go to cartesian:

$$r^2 = x^2 + y^2 + z^2$$

$$r\cos\theta = z$$

$$\Phi(x,y,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(x^2+y^2+z^2+d^2-2zd)^{1/2}} - \frac{1}{(x^2+y^2+z^2+d^2+2zd)^{1/2}}\right]$$

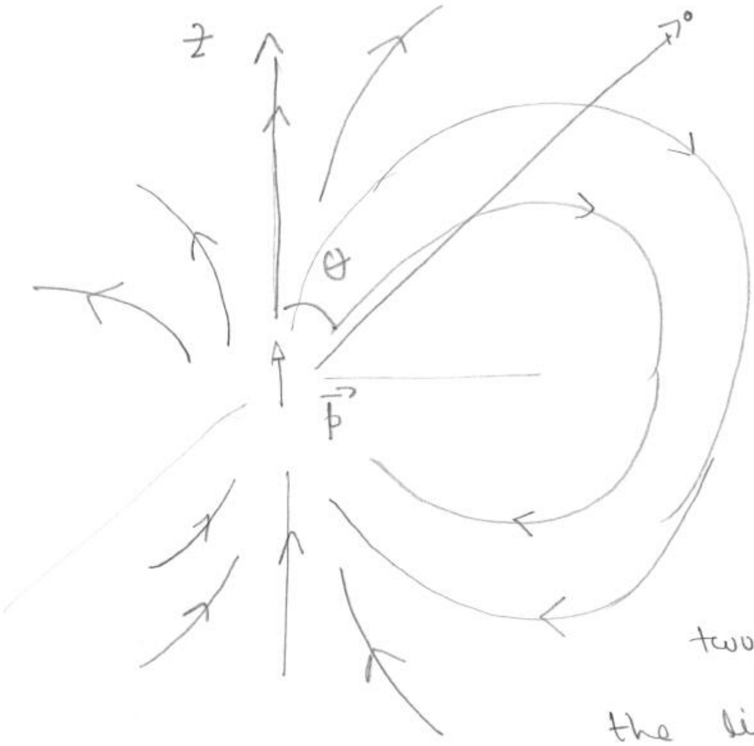
To get the electric field

$$\vec{E} = -\nabla\phi$$
$$= -\left(\hat{i}\,\partial_x + \hat{j}\,\partial_y + \hat{k}\,\partial_z\right)\phi(x,y,z)$$

This is a calculation left for problem set $\overline{\text{II}}$.

Another problem in the same problem set will be to calculate the surface charge density as a function of x and y on the surface of the conductor. Also calculate the total induced charge. Could you guess what that answer would be?

Field and potential of an electric dipole.



All the field lines are closed because there are no free charges.

Take the field of the two point charges and take the limit $d \to 0$ and $q \to \infty$ such that the product $\vec{p} = 2q\vec{d}$ remains constant.

$$\xi = \frac{d}{r}$$

$$\phi(r,\theta) = \frac{qd}{4\pi\epsilon_0}\frac{1}{d}\left[\frac{1}{\left(r^2+d^2-2dr\cos\theta\right)^{1/2}} - \frac{1}{\left(r^2+d^2+2dr\cos\theta\right)^{1/2}}\right]$$

$$= \frac{p}{4\pi\epsilon_0}\frac{1}{2rd}\left[\left(1+\xi^2-2\xi\cos\theta\right)^{-1/2} - \left(1+\xi^2+2\xi\cos\theta\right)^{-1/2}\right]$$

$$= \frac{p}{4\pi\epsilon_0}\frac{1}{2rd}\left[\cancel{1}-\cancel{\frac{\xi^2}{2}}+\xi\cos\theta-\left(\cancel{1}-\cancel{\frac{\xi^2}{2}}-\xi\cos\theta\right)+O(\xi^3)\right]$$

$$= \frac{p}{4\pi\epsilon_0}\frac{1}{2rd}\,2\xi\cos\theta = \frac{p}{4\pi\epsilon_0}\frac{1}{r\cancel{d}}\frac{\cancel{2}}{\cancel{2}}\frac{\cancel{d}}{r}\cos\theta + O(d^2)$$

$$\boxed{\begin{aligned}\phi(r,\theta) &= \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2} \\ &= \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}\end{aligned}}$$

comment

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{r}$$

$$\boxed{\phi(\vec{r}) = -\frac{1}{4\pi\epsilon_0}\,\vec{p}\cdot\vec{\nabla}\left(\frac{1}{r}\right)}$$

The potential of two point charges which are close to each other is such that the contribution from the two opposing charges would cancel but for their small difference in position. This difference essentially is $-\vec{\nabla}\left(\frac{1}{r}\right)\cdot\vec{d}$

This completes our story of how static charges interact, or various consequences of Coulomb's law. Tomorrow we shall start with how moving charges interact.

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