Lecture 2

Work and energy in Electrostatics

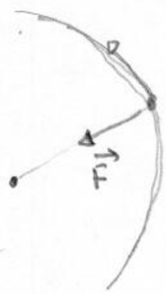
2.1 Work

In principle Coulomb's law is all we need to know in electrostatics. But the concepts of work and energy gives us both more power and deeper concepts.

From mechanics, you are already familiar with the concept of work. A force $\vec{F}$ acting on a body displaces it by an amount '$\vec{d}$' then the work done by the force is

$$W = \vec{F} \cdot \vec{d}$$

If the displacement is perpendicular to the force, obviously no work is being done. This is a case already familiar to you from uniform circular motion



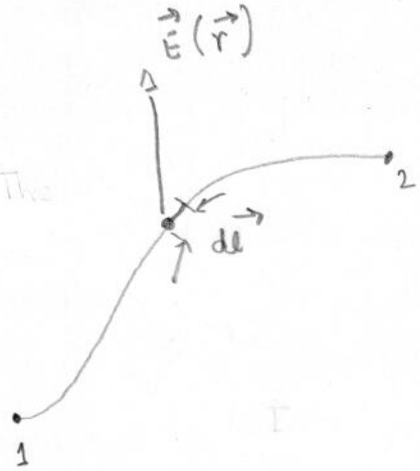
Rate of work done

$$\equiv \text{power} = \frac{dW}{dt}$$

$$= \vec{F} \cdot \vec{v} \quad (\text{for a constant in time } \vec{F})$$

$$= 0$$

We shall come back to this again when we deal with magnetic forces.



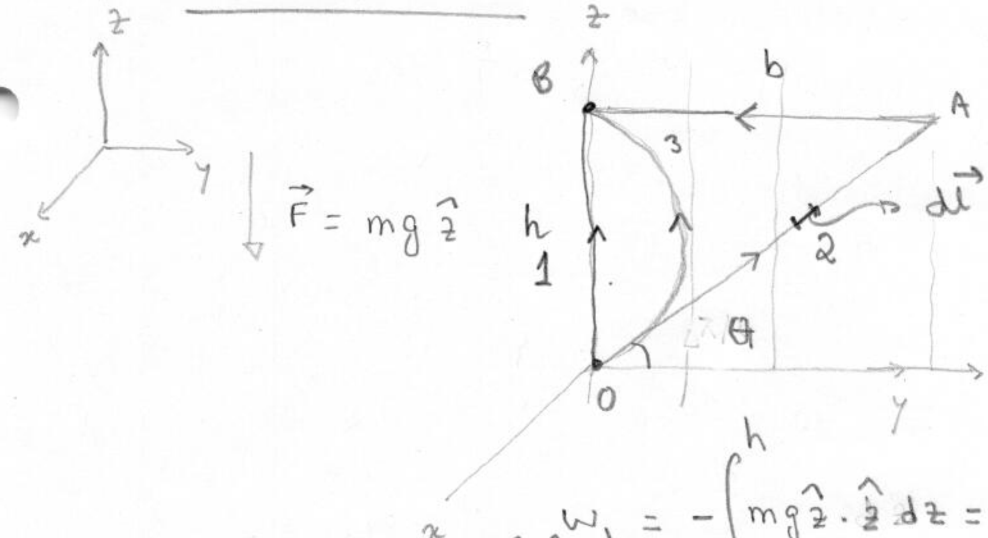
$$W(1\to 2) = \int_{\text{path}} \vec{F} \cdot d\vec{l}$$

line integral along a path.

In principle, the line integral of a vector field depends on the path you choose; not only the end points. But there are cases where it is not.

For example in the earth's gravitational field.

Example 2·1

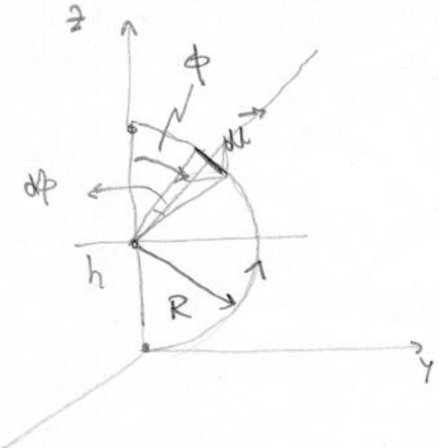


$$d\vec{l} = (\cos\theta\,\hat{y} + \sin\theta\,\hat{z})\,dl$$

$$W_1 = -\int_0^h mg\,\hat{z}\cdot\hat{z}\,dz = -mgh.$$

$$W_2 = -mg\sin\theta \int_0^{(h^2+b^2)^{1/2}} dl = -mg\sin\theta\,(h^2+b^2)^{1/2}$$

$$= -mg\frac{h}{(h^2+b^2)^{1/2}}(h^2+b^2)^{1/2} = -mgh.$$



$$\vec{dl} = R\,d\phi\left(\cos\phi\,\hat{z} + dl\,\sin\phi\,\hat{y}\right)$$

$$\int_{\pi} \vec{F}\cdot\vec{dl} = -R\,mg\int_0^{\pi}\cos\phi\,d\phi$$

$$= -mgR\,\sin\phi\Big|_0^{\pi}$$

$$= -2mgR = -mgh$$

2.2 Conservative fields and potential:

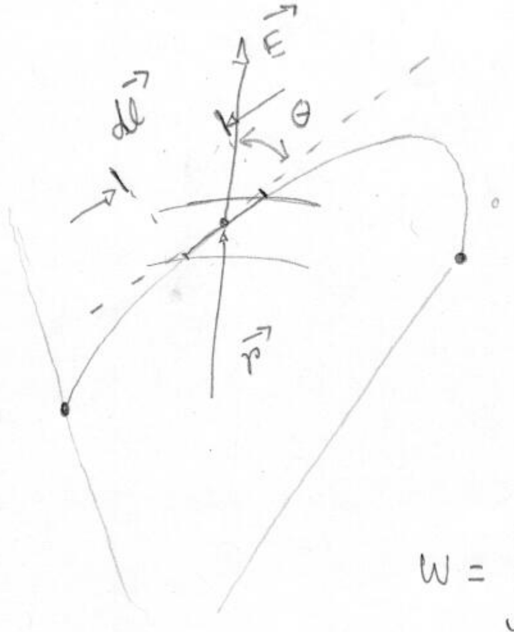
such a field for which the work done does not depend on path is called "conservative". This is a remarkable property which allows us to define the work done to dispalce an object by the two end point of path.

Let me first make a statement without proof:

$\boxed{\text{An electrostatic field is conservative.}}$

$$\Rightarrow \int_A^B \vec{E}\cdot\vec{dl} = \text{depends on A and B only.}$$

Proof for the field of a point charge:



$$dW = \vec{E} \cdot \vec{dl}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \cdot \left[\hat{r}\, dr + r\, d\theta\, \hat{\theta} + r \sin\theta\, d\phi\, \hat{\phi} \right]$$

$$W = \int_A^B \vec{E} \cdot \vec{dl}$$

$$= \frac{q}{4\pi\epsilon_0} \int_{r(A)}^{r(B)} \frac{1}{r^2}\, dr = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{r(B)} - \frac{1}{r(A)} \right]$$

Does not depend on path.

By the superposition principle the same should be true for many point charges.

$$W = \int_A^B \vec{E} \cdot \vec{dl}$$

$$= \int_A^B \left(\vec{E}_1 \cdot \vec{dl} + \vec{E}_2 \cdot \vec{dl} + \dots \right)$$

$$= \int_A^B \vec{E}_1 \cdot \vec{dl} + \int + \dots \int$$

Use a different coordinate system for each off them. Then the above proof of path independence would hold in each case.

comment

- The conservative nature is a property of the central force

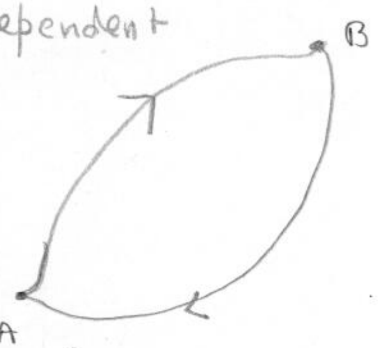
$$\vec{F} = f(r)\,\hat{r}$$

- All "fundamental" forces are conservative.

- Macroscopic effective forces, e.g., friction, air resistence are not conservative.

- If $\int_A^B \vec{E}\cdot d\vec{l}$ is path independent

$$\Rightarrow \boxed{\oint \vec{E}\cdot d\vec{l} = 0}$$



conservative $\Leftrightarrow \oint \vec{E}\cdot d\vec{l} = 0$

- At every point in an electric field an unique (upto a fixed reference value) potential can be defined

$$\phi(\vec{r}) = -\int^{\vec{r}} \vec{E}\cdot d\vec{l}$$

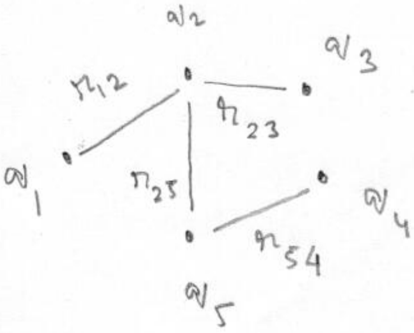
ref point

over any path

potential.

convention

2·3 Energy in assembling a collection of point charges.



Energy in bringing $q_1 = 0$

bringing q_2 in the presence of q_1 $= \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}}$

bringing q_3 - - - $\{ q_1, q_2 \}$ $= \frac{1}{4\pi\epsilon_0} \left[\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right]$

⋮

$$\text{Total} = \frac{1}{4\pi\epsilon_0} \Big[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \cdots + \frac{q_1 q_j}{r_{1j}}$$

$$+ \quad \frac{q_2 q_3}{r_{23}} + \cdots + \frac{q_2 q_j}{r_{2j}}$$

$$+ \quad + \frac{q_3 q_4}{r_{34}} + \cdots + \frac{q_3 q_j}{r_{3j}}$$

$$+ \; - - - \quad \Big]$$

$$\text{Total} = -\frac{1}{4\pi\epsilon_0}\left[\sum_{j=2}^{N}\frac{q_1 q_j}{r_{1j}} + \sum_{j=3}^{N} + \cdots\right]$$

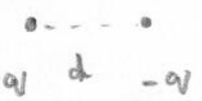
$$= -\frac{1}{4\pi\epsilon_0}\sum_{k=1}^{N}\sum_{j=k+1}^{N}\frac{q_k q_j}{r_{kj}}$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{1}{2}\sum_{k=1}^{N}\sum_{j\neq k}\frac{q_k q_j}{r_{kj}}$$

= Energy stored in the system of charges.

<u>Example 2.2</u>

Enery of a dipole



$$\text{Energy} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{d}$$

dipole moment = qd

Example 2.3

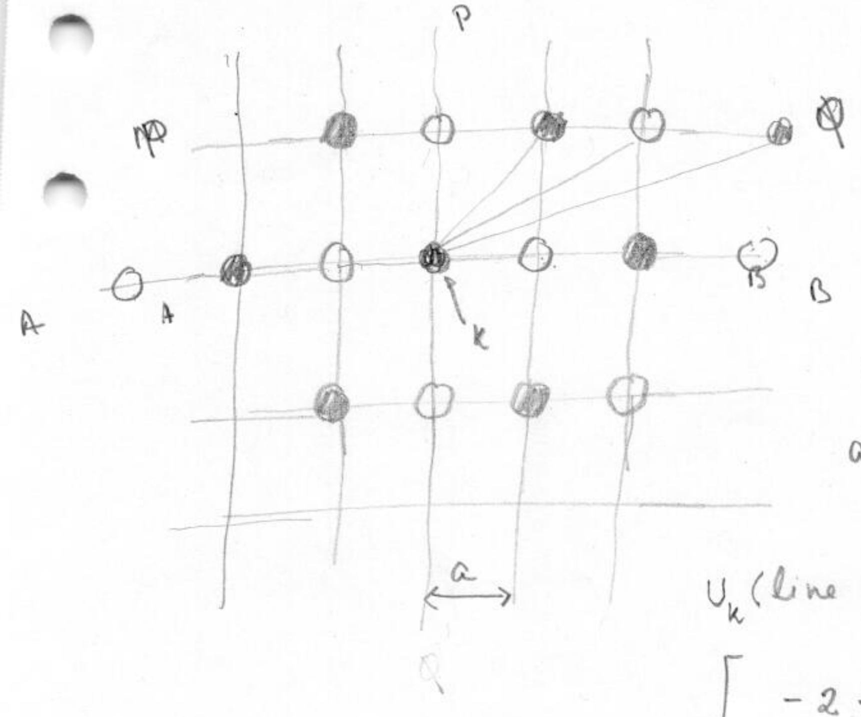
Energy of an ionic crystal :

If we think of applying formula of energy of a collection of charges to a real physical system the double sum looks fantastic. On one hand enery of a macroscopic system, intutively speaking, should be proportional to its volume such that the concept of energy per unit volume makes sense. On the other hand the double sum contains N^2 number of terms if there are N charges.

To see how this two ideas can be reconciled let us consider a common ionic crystal NaCl.

The structure is known from X-ray diffraction and is a simple cubic 3-d lattice.

Let us consider a plane



$$U = \frac{1}{4\pi\epsilon_0} \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{q_i q_j}{r_{ij}}$$

First sum of charge k along the x axis

$$U_k(\text{line } AB) = \frac{1}{4\pi\epsilon_0} \frac{1}{2} \frac{e^2}{a} \left[-2 + \frac{2}{2} - \frac{2}{3} + \frac{2}{4} - \frac{2}{5} + \cdots \right]$$

$$U_k(\text{line } AB) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{a}\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\right]$$

consider the series

$$\ln(1+x) = \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right]$$

at $x = 1$,

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$U_k(\text{lin } AB) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{a}\ln 2 \qquad \ln 2 \sim 0{\cdot}693$$

$$U_1 = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{a}\ln 2$$

$$U_k(\text{line } MN) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{a}\left[1 - \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{5}} - \frac{2}{\sqrt{10}} + \cdots\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{e^2}{a}\left[1 - \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{5}} - \frac{2}{\sqrt{10}} + \cdots\right]$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{e^2}{a}\left[1 - 2\sum_{k=1}^{N}\frac{(-1)^k}{(1+k^2)^{1/2}}\right]$$

There are 4 such lines

$$U_2 = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{a}\cdot 4\cdot\left[1 - 2\sum_{k=1}^{N}\frac{(-1)^k}{(1+k^2)^{1/2}}\right]$$

$$\frac{1}{4\pi\epsilon_0}\frac{e^2}{a}$$

$a \sim 2.81\,\mathring{A} \approx 2.8\times10^{-10}$ m

$e \sim 1.6\times10^{-19}$ C

$$\simeq 9\times\cancel{10}^{9} \cdot \frac{1.6\times\cancel{10}^{-19}}{2.8\times10^{-10}} \quad eV$$

$$\simeq \frac{9\times1.6}{2.8} \; eV$$

$$\sim 5.12 \; eV$$

This energy is equal to the energy of vaporization of NaCl + the energy required to make ions from NaCl molecule. Experimentally this is known to be about 7.92 eV per molecule.

The above energy we calculate is per ion and two ions make a molecule, so

$$U\,(\text{per molecule}) = \frac{1}{4\pi\epsilon_0}\left(\cancel{\tfrac{1}{2}}\right)\left(\frac{1}{N}\right)\sum_{\substack{i,j=1 \\ i\neq j}}^{N}\frac{q_i q_j}{r_{ij}}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{e^2}{a}\frac{1}{N}\;N\left[\,U_1 + 4U_2 + \cdots\,\right]$$

Remarkably, as each of $U_1, U_2, \ldots$ converges the double sum becomes extensive ($\sim N$) hence we can define energy per molecule.

crudest approximation

$$U \text{ (per molecule) } \sim -5.12 \text{ eV } (\underbrace{2 \ln 2}_{1.386})$$

$$\sim -7.09 \text{ eV}$$

Not too bad.

A more involved calculation gives

$$U \text{ (per molecule) } \sim -5.12 \text{ eV } (1.747)$$

$$\sim -8.94 \text{ eV}$$

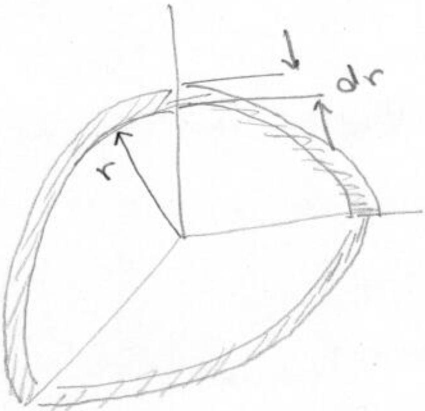
which is somewhat more than the expected value. Furthermore the sign is negative and becomes more negative as 'a' becomes smaller without bound.

$\Rightarrow$ Electrostatically ions should not be stable but collapse!

- There is actually a repulsive force as the ions are quashed together.
- We have also ignored the kinetic energy of the lattice.

Example 2.4

Energy stored in a charged sphere of radius 'a' filled with uniform charge density 'ρ'



$$dU = \frac{1}{4\pi\epsilon_0} (4\pi r^2) dr\, \rho\, \frac{4}{3}\pi r^3 \rho \frac{1}{r}$$

$$U = \frac{1}{4\pi\epsilon_0} \rho^2 (4\pi)\left(\frac{4}{3}\pi\right) \int_0^a r^4 dr$$

$$= \frac{1}{4\pi\epsilon_0} (4\pi)\left(\frac{4}{3}\pi\right) \rho^2 \frac{a^5}{5}$$

$$= \frac{1}{4\pi\epsilon_0} \left(\frac{4}{3}\pi a^3 \rho\right)\left(\frac{4}{3}\pi a^3 \rho\right)\left(\frac{1}{a}\right) \frac{1}{5}\ 3$$

$$= \frac{1}{4\pi\epsilon_0} \left(\frac{Q^2}{a}\right)\left(\frac{3}{5}\right) = \frac{3}{5}\left(\frac{Q^2}{4\pi\epsilon_0 a}\right)$$

comment

- It blows up as the sphere gets smaller for a fixed Q. In other words, the energy necessary to 'make' a point charge is 'infinite' !!

2.4 Potential difference and potential.

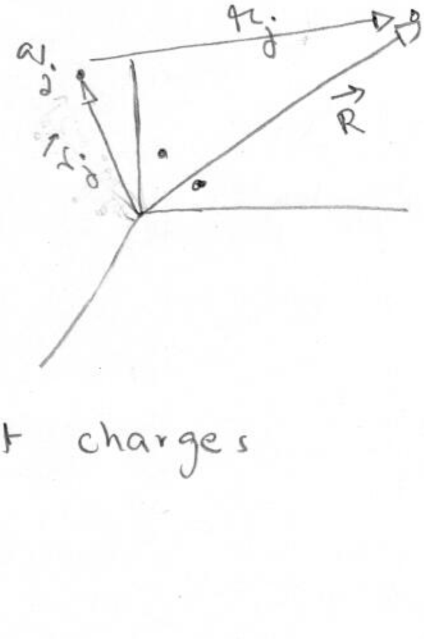
potential difference in an electric field is the work done per unit (test) charge to move it from A to B

$$\phi(B) - \phi(A) = -\int_A^B \vec{E} \cdot d\vec{l}$$

This does not depend on path.

so if we choose the potential at infinity to be zero, we can define

$$\phi(\vec{R}) = -\int_\infty^{\vec{R}} \vec{E} \cdot d\vec{l}$$



For a point charge

$$\phi(\vec{R}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

for a collection of point charges

$$\phi(\vec{R}) = \frac{1}{4\pi\epsilon_0} \sum_j \frac{q_j}{r_j}$$

For a continuous charge distribution

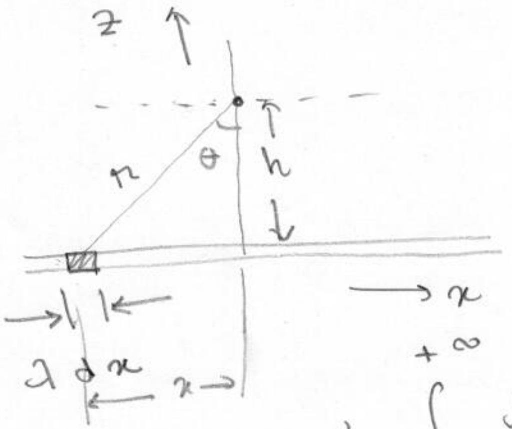
$$\phi(\vec{R}) = \frac{1}{4\pi\epsilon_0} \int_{\text{source}} \frac{\rho \, dV}{r}$$

Comment:

- potential is easier to calculate than the field because it is a scalar.
- But it falls off slower at larger distances.

Example 2.5

potential of an infinite line charge



$$\phi(R) = \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{\lambda\, dx}{r} \qquad r^2 = x^2 + h^2$$

$$= \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{dx}{\left(x^2+h^2\right)^{1/2}} \qquad \frac{x}{h} = \xi \quad dx = h\, d\xi$$

$$= \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{h\, d\xi}{h\left(\xi^2+1\right)^{1/2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{d\xi}{\sqrt{\xi^2+1}}$$

$\longrightarrow$ does not converge!

From example 1·6 we know that

$$\vec{E}(\vec{R}) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{r}\hat{r}$$

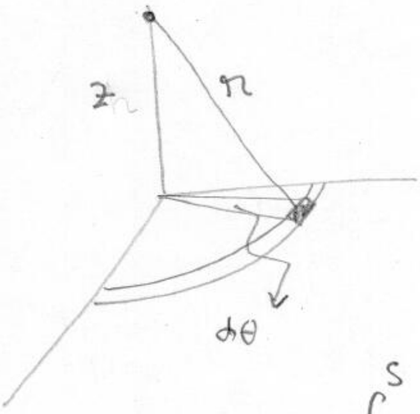
$$\phi(r_2) - \phi(r_1) = -\int_{r_1}^{r_2} \vec{E}\cdot d\vec{l}$$

$$= -\frac{1}{4\pi\epsilon_0} 2\lambda \left(\ln r_2 - \ln r_1\right)$$

Blows up as r_1 goes to infinity.

$\Rightarrow$ potential at infinity is not zero because the charge distribution goes to infinity.

Example 2·6



potential at the axis of a disk

$$ɼ^2 = h^2 + r^2$$

$$V(R) = \frac{1}{4\pi\epsilon_0}\int_{disk}\frac{\sigma\, r\, d\theta\, dr}{ɼ}$$

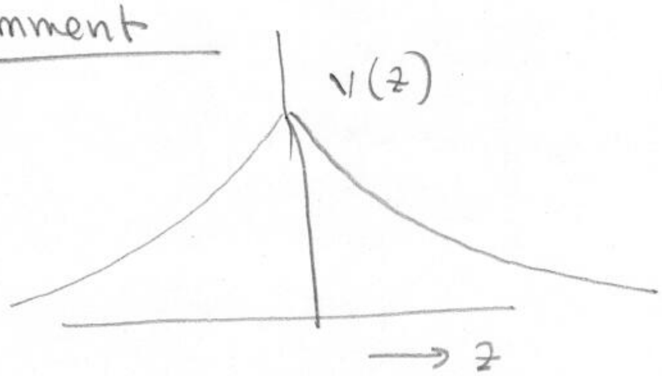
$$= \frac{1}{4\pi\epsilon_0}\int_0^S \frac{\sigma\, r\, dr}{\left(z^2+r^2\right)^{1/2}}\int_0^{2\pi} d\theta$$

$$Q = \pi S^2 \sigma$$

$$= \frac{1}{4\pi\epsilon_0} 2\pi\sigma\left[\sqrt{S^2+z^2} - z\right]$$

$$= \frac{1}{4\pi\epsilon_0} 2.\ Q\left[\sqrt{1+\frac{z^2}{S^2}} - z\right]$$

comment



- different sign of z for +ve and -ve z

- At large z

$$(s^2+z^2)^{1/2} \simeq z\left(1+\xi^2\right)^{1/2} \qquad \xi = \frac{s}{z} << 1$$

$$\simeq z\left(1+\frac{\xi^2}{2}+\dots\right)$$

$$(s^2+z^2)^{1/2} - z$$

$$\simeq z\left(1+\frac{\xi^2}{2}+\dots\right) - z$$

$$\simeq \frac{1}{2}\xi^2 z + \dots$$

$$\simeq \frac{1}{2}\frac{s^2}{z}$$

$$V(z) \sim \frac{1}{4\pi\epsilon_0}\,\cancel{2}\pi\sigma\,\frac{1}{\cancel{2}}\frac{s^2}{z}$$

$$\sim \frac{1}{4\pi\epsilon_0}\frac{Q}{z}$$ ~ monopole contribution.

2.5 Gradient of a scalar.

The potential $\phi(x,y,z)$ can be considered a function of space once the an additive constant is given. Hence we can ask, how does potential changes as we move position a little bit!

$$\phi(\vec{r}+d\vec{r}) = \phi(\vec{r}) + \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) + \cdots \quad \text{taylor series}$$

$$= \phi(\vec{r}) + \vec{\nabla}\phi \cdot d\vec{r} + \cdots$$

formally

$$\vec{\nabla} = \left(\hat{i}\,\partial_x + \hat{j}\,\partial_y + \hat{k}\,\partial_z\right)$$

the same operator we met in last lecture.

But we also know that

$$\phi(\vec{r} + d\vec{r}) - \phi(\vec{r}) = - \vec{E} \cdot d\vec{r}$$

$$\Rightarrow \quad \boxed{\vec{E} = - \nabla \phi}$$

2·6 Laplacian.

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \quad \vec{\nabla} \cdot (\nabla \phi) = - \frac{\rho}{\epsilon_0}$$

$$\left(\hat{i}\, \partial_x + \hat{j}\, \partial_y + \hat{k}\, \partial_z \right) \cdot \left(\hat{i}\, \partial_x \phi + \hat{j}\, \partial_y \phi + \hat{k}\, \partial_z \phi \right) = - \frac{\rho}{\epsilon_0}$$

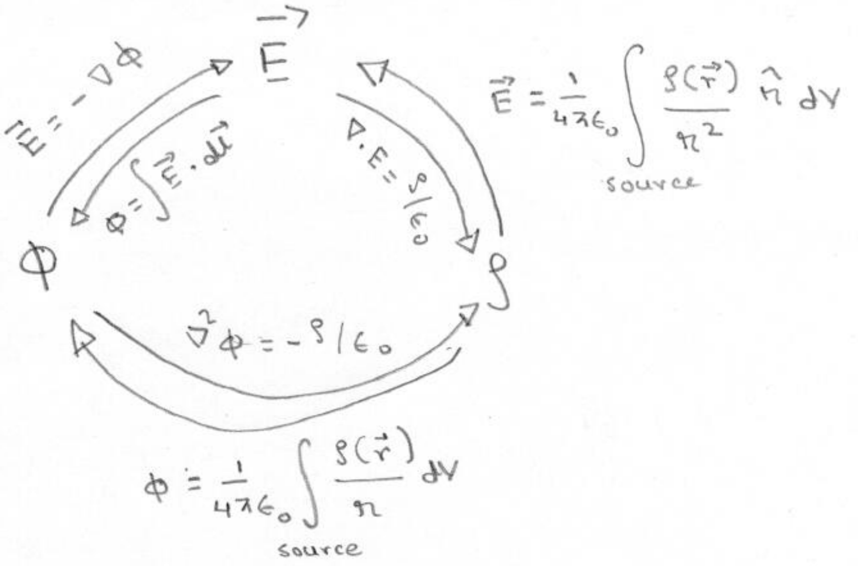
$$\Rightarrow \quad \left(\partial_x^2 + \partial_y^2 + \partial_z^2 \right) \phi = - \frac{\rho}{\epsilon_0}$$

$$\boxed{\nabla^2 \phi = - \frac{\rho}{\epsilon_0}}$$ Poisson's eqn

In charge-free space

$$\boxed{\nabla^2 \phi = 0}$$ Laplace's eqn.

2·7 Summary of electrostatics



2·8 The gradient

consider a scalar field, e.g. temperature. If we want to describe it's variations in space, we need to specify its gradient everywhere. The gradient of temperature determines the heat flow. The heat flow vector

$$\vec{h} = -\nabla T$$

Comment

- Is gradient a vector ?

Clearly ΔT between two neighbouring points is a physical invariant

$$\underbrace{\Delta T}_{\text{scalar}} = \underbrace{(\nabla T)}_{?} \cdot \underbrace{(d\vec{r})}_{\text{vectror}}$$

dot product

Hence ∇T must be vector.

- Equi-ϕ surfaces; equipotential, isothermals.

Along an equi-ϕ surface

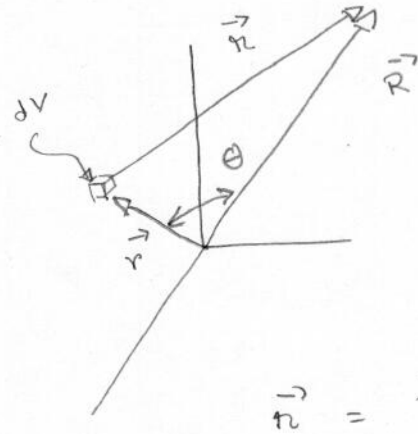
$$\begin{aligned} \Delta \phi &= 0 \\ &= (\nabla \phi) \cdot \left(\text{tangent to the surface}\right) \\ &= 0 \end{aligned}$$

$\Rightarrow$ $\nabla \phi$ is along the normal to equi-ϕ surfaces.

- Field lines and equipotentials must be normal to each other at every point in space.

2.9 (extra material)

multipole expansion



$$\phi(\vec{R}) = \frac{1}{4\pi\epsilon_0} \int_{\text{source}} \frac{\rho(\vec{r})\, dV}{r}$$

$$\vec{r} + \vec{r} = \vec{R}$$

$$\vec{r} = \vec{R} - \vec{r}$$

$$r^2 = R^2 + r^2 - 2Rr\cos\theta$$

$$\phi(\vec{R}) = \frac{1}{4\pi\epsilon_0} \int_{\text{source}} \frac{\rho(\vec{r})\, dV}{\left(R^2 + r^2 - 2Rr\cos\theta\right)^{1/2}}$$

$$r = \left(R^2 + r^2 - 2Rr\cos\theta\right)^{1/2}$$

$$= R\left[1 + \frac{r^2}{R^2} - 2\frac{r}{R}\cos\theta\right]^{1/2}$$

$$=$$

$$\frac{1}{r} = \frac{1}{R}\left[1 + \left(\frac{r^2}{R^2} - 2\frac{r}{R}\cos\theta\right)\right]^{-1/2}$$

$$\left(1+\xi\right)^{-1/2} = 1 - \frac{1}{2}\xi + \frac{3}{8}\xi^2 + \cdots$$

$$-\frac{1}{2}\left(\frac{r^2}{R^2} - 2\frac{r}{R}\cos\theta\right)$$

$$+\frac{3}{8}\left(\frac{r^2}{R^2} - 2\frac{r}{R}\cos\theta\right)^2$$

$$+ \cdots$$

$$= + \frac{r}{R}\cos\theta - \frac{1}{2}\frac{r^2}{R^2} + \frac{3}{8}.4.\frac{r^2}{R^2}\cos^2\theta + \cdots$$

$$= \frac{r}{R}\cos\theta + \frac{r^2}{R^2}\left\{\frac{3}{2}\cos^2\theta - \frac{1}{2}\right\} + \cdots$$

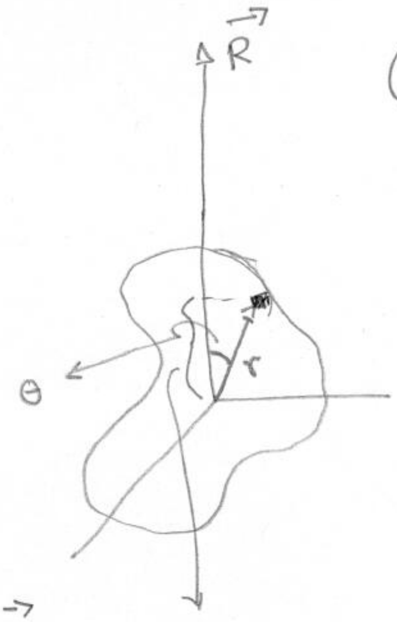
$$\Rightarrow \quad \Phi(R) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{R}\int_{\text{source}} \rho \, dV + \frac{1}{R^2}\int \rho \, r\cos\theta \, dV\right.$$

$$+ \frac{1}{R^3}\int \rho \, r^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) dV$$

$$\left. + \cdots \right]$$

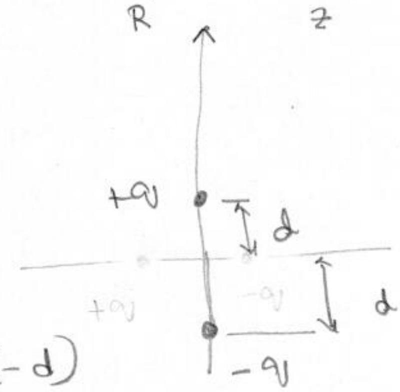
$$= \frac{1}{4\pi\epsilon_0}\left[\frac{Q_0}{R} + \frac{Q_1}{R^2} + \frac{Q_2}{R^3} + \cdots\right]$$

$$\Phi_1 = \int \rho\, r \cos\theta \, dv$$



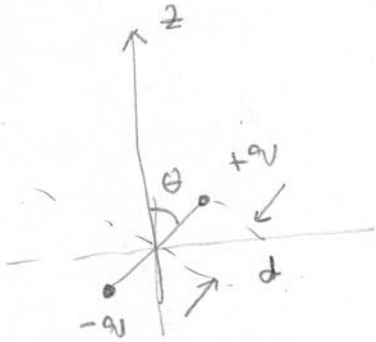
$$= \int \rho\, z \, dV$$

$= 0$ for a spherically symmetric distribution

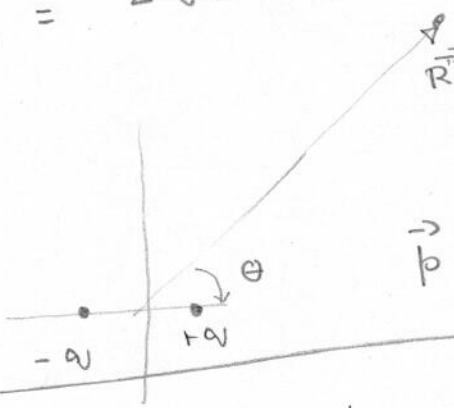


$$\Phi_1 = q d + (-q)(-d)$$

$$= 2 q d$$



$$\Phi_1 = 2 q d \cos\theta$$



$$\vec{p} = 2 q \vec{d}$$

$$\boxed{\Phi_1(\vec{R}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\hat{R}}{R^2}}$$

potential of a dipole

$$E_1(\vec{R}) = -\nabla\Phi$$

$$\sim \frac{1}{R^3}$$