1. Electrostatics

1.1 Electric charges, particle and anti-particle

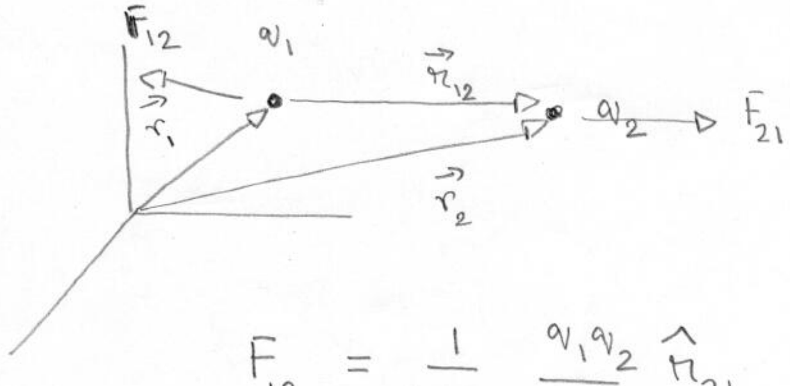
1.2 Charge is conserved

1.3 charge is quantized

There are no free quarks (charge $1/3\ e$)

In condensed matter physics certain experiements show that charge can be carried in units of fractional e but such particles are not "elementary particles" but "effective particles.

1.4 Coulomb's law



$$F_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{21}^2} \hat{r}_{21}$$

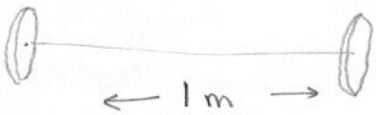
Comment: corret only for static charges.

$$\epsilon_0 = 8\cdot 85 \times 10^{-12} \ \frac{\text{coul}^2}{\text{N m}^2}$$

permitivity of free space

$$\frac{1}{4\pi \epsilon_0} \approx 8\cdot 988 \times 10^{9} \ \text{SI units}$$

Example 1.1



A 10 Kr coin weighs 6·6 gm

They are placed 1 m apart.

Normally they are charge neutral. But assume that somehow each has acquired a charge of 1 coulomb. The electrostatic force will be:

$$F_{\text{coulomb}} \simeq 9 \times 10^{9} \ \text{Newton} \ !$$

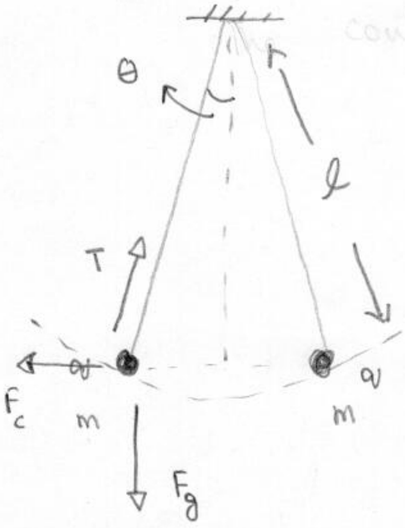
The gravitational force between them will be

$$F = G \frac{(6\cdot 6 \text{ gm})^2}{(1 \text{ m})^2} = (6\cdot 6)^2 \times 10^{-6} \ \frac{G \text{ Kg}^2}{\text{m}^2}$$

$$\simeq 42 \times 10^{-6} \ 6\cdot 6 \times 10^{-11} \ \text{N}$$

$$\simeq 10^{-14} \ \text{N}$$

Example 1.2



$$F_c = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2l\sin\theta)^2}$$

$$F_g = mg$$

$$T\sin\theta = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2l\sin\theta)^2}$$

$$T\cos\theta = mg$$

$$\tan\theta = \frac{1}{4\pi\epsilon_0}\frac{q^2}{4l^2\sin^2\theta}\frac{1}{mg}$$

$l \sim 1$ m

$m \sim 1$ gm $= 10^{-3}$ kg

$$\tan\theta\sin^2\theta = 9\times10^{9}\frac{q^2}{4\,.\,1.}$$

Assume θ is small

$\Rightarrow \tan\theta \sim \theta, \quad \sin\theta \sim \theta$

$$\Rightarrow \theta^3 \sim 9\times10^{9}\frac{q^2}{4}$$

$\Rightarrow$ To make the small θ approximation $\theta \sim 0.1$

$$10^{-3} \sim 9\times10^{9}\frac{q^2}{4} \Rightarrow \boxed{q \sim \frac{2}{3}10^{-6}\ C}$$

$\Rightarrow$ To get forces of the same order as terrestrial gravity we need to deal with charges of order μ Coulomb

1·5 Principle of superposition

Force on q_1 due to q_2, and q_3 is the sum of their individual forces

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13}$$

$\Rightarrow$ All interactions are pair-wise

1·6 Electric field

For a certain distribution of source charge's (q_j) calculate the force on a test charge Q at $\vec{R}$

Then the electric field at $\vec{R}$ is

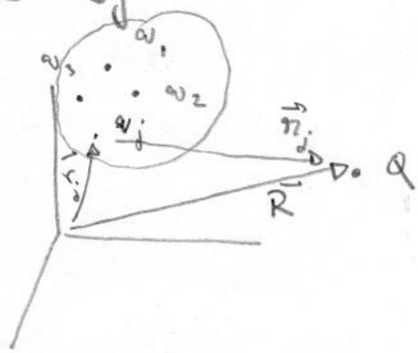
$$\vec{E}(\vec{R}) = \lim_{Q \to 0} \frac{\vec{F}(Q)}{Q}$$

comment

1. The rigor implied by the limit is false because we know that in practice charge is quantized.

Better to define electric field by

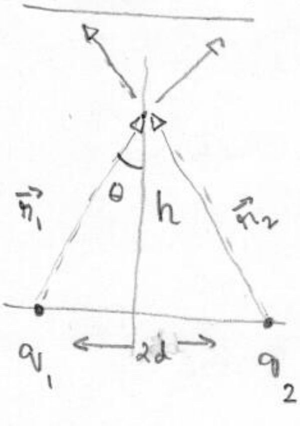
$$\vec{E}(R) = \sum_j \frac{q_j}{r_j^2}\,\hat{r}_j\,\frac{1}{4\pi\epsilon_0}$$



2. Electric field is a local concept.

3. Is electric field real ?

Example 1.3



$$\vec{E} = \left(\hat{r}_1 \frac{q_1}{r_1^2} + \hat{r}_2 \frac{q_2}{r_2^2}\right)\frac{1}{4\pi\epsilon_0}$$

$$r_1 = \left(h^2 + d^2\right)^{1/2}$$

$$\hat{r}_1 = \left(-\hat{x}\,d + \hat{z}\,h\right)\frac{1}{\left(h^2+d^2\right)^{1/2}}$$

$$r_2 = r_1$$

$$\hat{r}_2 = \left(\hat{x}\,d + \hat{z}\,h\right)\frac{1}{\left(h^2+d^2\right)^{1/2}}$$

$$\vec{E} = \frac{2q}{4\pi\epsilon_0}\frac{h\,\hat{z}}{\left(h^2+d^2\right)^{3/2}}, \quad q_1 = q_2$$

For large h $\quad E \approx \hat{z} \frac{2q}{4\pi\epsilon_0} \frac{h}{h^3} \sim \hat{z} \frac{1}{4\pi\epsilon_0} \frac{2q}{h^2}$

At large distance from a collection of point charges $\quad E \sim \underbrace{\frac{1}{4\pi\epsilon_0}\frac{\sum q_j}{r^2}}_{\text{monopole.}} + \cdots$

For $\quad q_2 = -q_1$

$$\underline{E} = \frac{1}{4\pi\epsilon_0} \hat{x} \frac{2qd}{(h^2+d^2)^{3/2}}$$

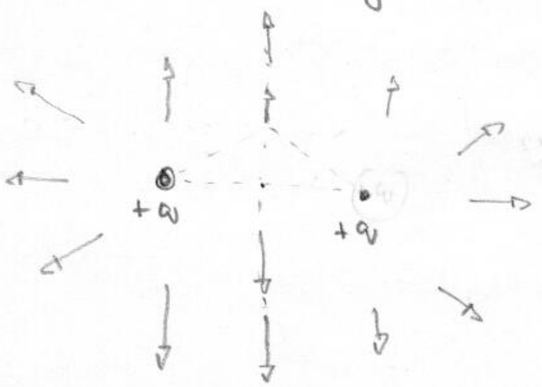
For large h

$$\vec{E} \approx \frac{1}{4\pi\epsilon_0} \frac{\vec{p}}{h^3} \sim \frac{1}{h^3} \quad \text{(not inverse-square!)}$$

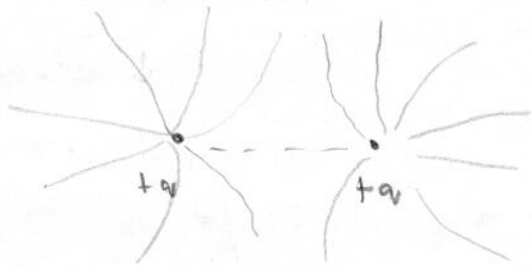
$\vec{p} = q\,(2d\hat{x})$ dipole moment

The monopole contribution is zero because the net charge at source is zero.

1.7 visualization of electric field



arrows



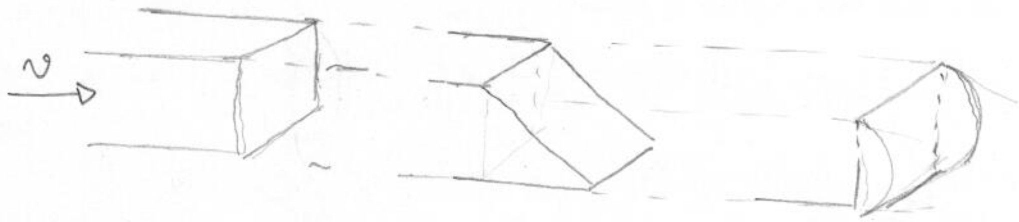
lines of force.

For 3d, rotate about the axis of symmetry.

comment lines of forces are not the trajectory of unit test charge.

1.8 Flux

- How much water flows through the following areas?

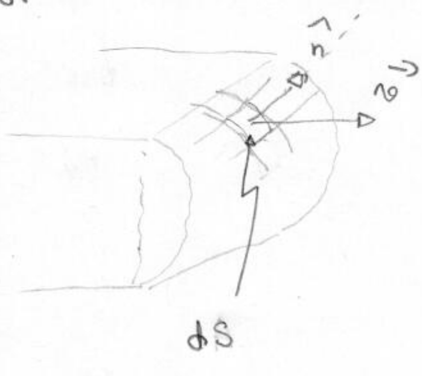


$$\text{flux } \Phi = \vec{v} \cdot \vec{A}$$

$$= v A \cos\theta$$

remains constant over the first two surfaces.

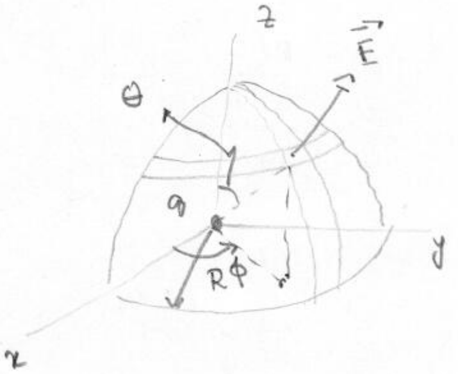
For the last one, consider:



$$\Phi = \int_S \vec{v} \cdot \hat{n} \, ds$$

Example 1·3

Flux of the electric field due to a point charge on the surface of a sphere.



$$dS = R^2 \sin\theta \, d\theta \, d\phi$$

$$\hat{n} = \hat{r}$$

$$\vec{E} = \frac{q}{R^2} \hat{r} \quad \frac{1}{4\pi\epsilon_0}$$

$$\Phi = \oint_S \vec{E} \cdot \hat{n} \, ds = \frac{Q}{R^2} R^2 \oint \sin\theta \, d\theta \, d\phi \left(\frac{1}{4\pi\epsilon_0}\right)$$

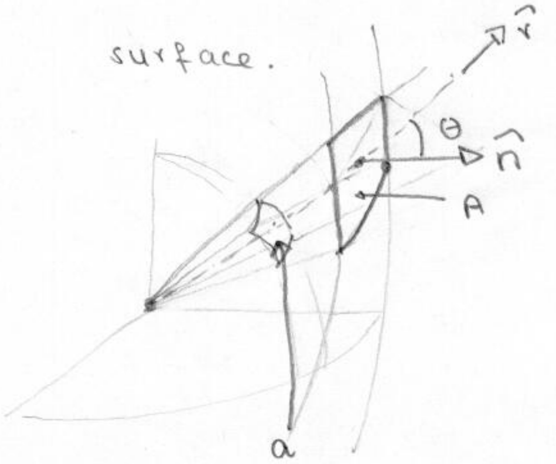
closed surface

$$= Q \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \quad \frac{1}{4\pi\epsilon_0}$$

$$\boxed{\Phi = \frac{Q}{\epsilon_0}}$$

Example 1.4

same as 1.3 but over an arbitrary (smooth) surface.

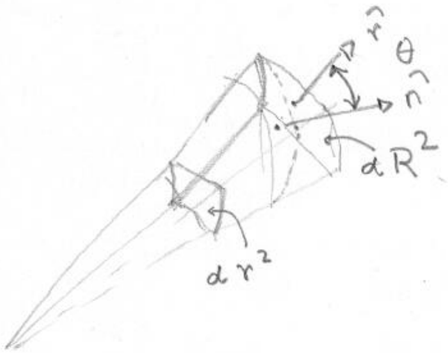


The outer area can be thought of as surface element of a bigger sphere (radius R) projected by θ.

Flux through outer patch : $\vec{E}_{(R)} \cdot \hat{n}\, A$

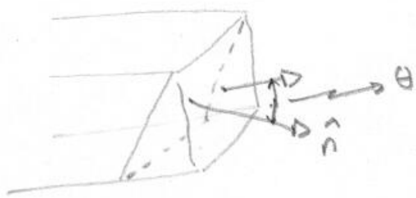
$$\Phi_{outer} = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{R^2}\right) A \cos\theta$$

Flux through inner patch



$$\Phi_{inner} = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r^2}\right) a$$

$$\left(\frac{A}{a}\right) = \left(\frac{R}{r}\right)^2 \frac{1}{\cos\theta}$$



Ratio of the fluxes

$$\frac{\Phi_{outer}}{\Phi_{inner}} = \frac{A\cos\theta}{R^2} \cdot \frac{r^2}{a} = 1.$$

$$\boxed{\oint \vec{E}\cdot\hat{n}\,ds = \frac{1}{\epsilon_0} Q_{enc}}$$ Gauss's law

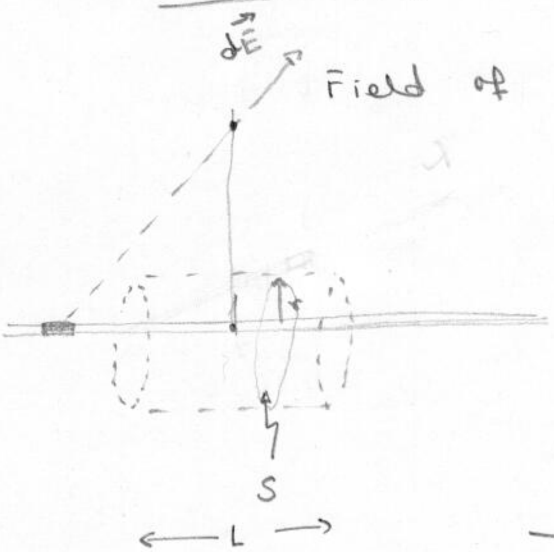
what happens if Coulomb's law is replaced by inverse cube law?

1.9 Application of Gauss's law and symmetry

Example 1.5

Field of a spherical charge distribution.

Example 1.6



Field of a line charge

$$\oint \vec{E}\cdot\hat{n}\,ds$$

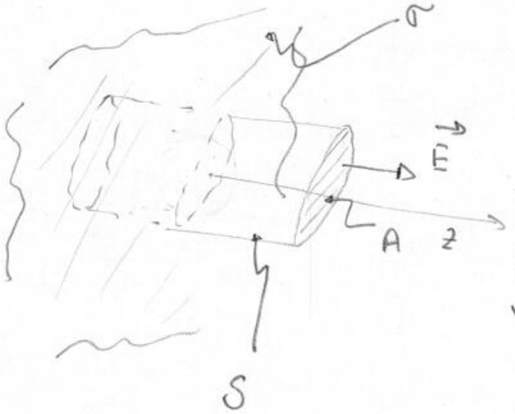
$$= (2\pi r L)\, E(r)$$

$$= Q_{enc} = \lambda L \frac{1}{\epsilon_0}$$

$$\Rightarrow \boxed{\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{r}\hat{r}}$$

does not fall off as $\frac{1}{r^2}$ at large r!

Example 1.7



Infinite plane with surface charge density σ.

$$\oint \vec{E} \cdot \hat{n}\, ds$$

$$= 2\, E(z)\, A$$

$$= Q_{enc} \frac{1}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

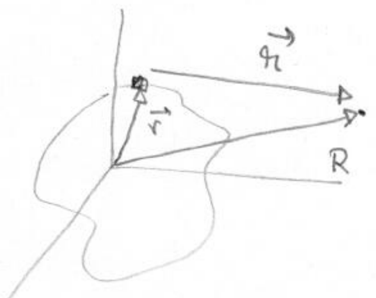
$$\Rightarrow \boxed{\begin{aligned} E(z) &= \frac{1}{2\epsilon_0} \sigma \\ &= \frac{1}{4\pi\epsilon_0} 2\pi\sigma \end{aligned}}$$

Does not depend on z at all !

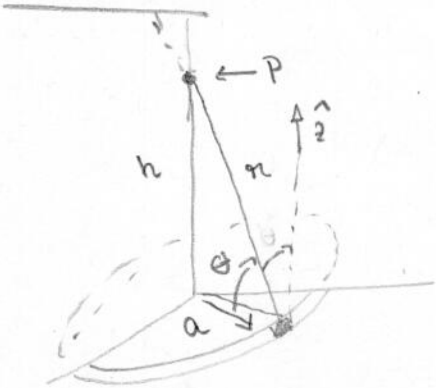
1.10 From discrete to continious charge distribution.

$$\vec{E}(\vec{R}) = \int_{\text{source}} \frac{\rho(\vec{r})\, dV}{4\pi\epsilon_0 \; \mathscr{r}^2} \hat{\mathscr{r}}$$

$$\vec{\mathscr{r}} = \vec{R} - \vec{r}$$



Example 1.8



$$E(P) = \int_{\text{source}} \frac{\lambda\, a\, d\phi}{r^2}\, \hat{r}\, \frac{1}{4\pi\epsilon_0}$$

$$r^2 = a^2 + h^2$$

$$\hat{r} = \sin\theta\, \hat{z} - \cos\theta\, \hat{a}$$

$$\cos\theta = \frac{a}{r} = \frac{a}{(a^2+h^2)^{1/2}}$$

$$E(P) = \lambda a \frac{1}{4\pi\epsilon_0}\left[\int_0^{2\pi} \hat{z}\, \frac{\sin\theta\, d\phi}{r^2} - \int_0^{2\pi} \hat{a}\, \frac{\cos\theta\, d\phi}{r^2}\right]$$

$$= \frac{\lambda a}{4\pi\epsilon_0}\left[\frac{h}{(a^2+h^2)^{3/2}}\, \hat{z} \int_0^{2\pi} d\phi - \frac{\cos\theta}{r^2}\int_0^{2\pi} d\phi\, \hat{a}\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\, \frac{h}{(a^2+h^2)^{3/2}}$$

$\sim \frac{1}{h^2}$ for large h

$= 0$ for $h = 0$

1.11 local conservation laws

charge inside this volume is

$$Q = \int_V \rho \, dV$$

Q is a conserved quantity.

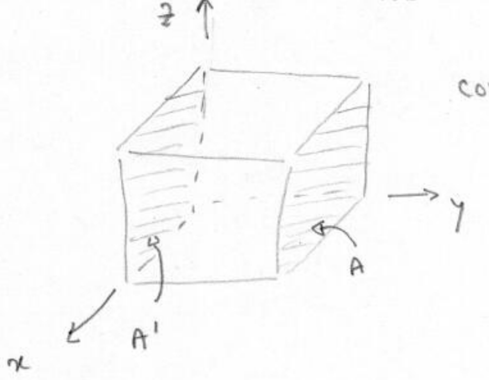
$\Rightarrow$ Q does not change with time.

$$\frac{dQ}{dt} = 0$$

except if charges enter or exit this volume.

The entry or exit is given by the flux of charged matter through the surface S

To make it simple, consider V to be a box.



consider the two faces, A and A'

The rate of flow of charge through A is

$$- A v_y(A) \rho(A)$$

flowing out of the box

For a general volume V

$$\frac{dQ}{dt} = - \oint_S \rho \vec{v} \cdot \hat{n} \, dS$$

Hence charge conservation implines that

$$\int_V (\partial_t \rho)\, dV = - \oint \rho \vec{v} \cdot \hat{n}\, ds$$

Identify the current density

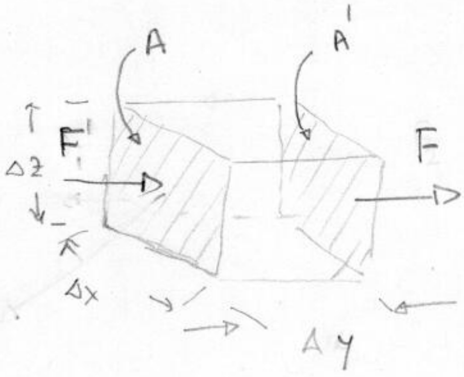
$$\vec{J} = \rho \vec{v}$$

$$\Rightarrow \int_V (\partial_t \rho)\, dV = - \oint_S \vec{J} \cdot \hat{n}\, ds$$

1.12 Flux over infinitismal volume.

Consider a general vector field $\vec{F}$

Let us calculate it's flux on a very small box



First along the y direction.

Flux through A

$$\vec{F}(A) \cdot (-\hat{y})\, \Delta x\, \Delta z$$

$$= - F_y(A)\, \Delta x\, \Delta z$$

Flux through A' $= F_y(A')\, \Delta x\, \Delta z$

Net flux in the y direction through this box

$$\Phi_y = \left[F_y(A') - F_y(A) \right] \Delta x\, \Delta z$$

$A \rightarrow (x, y, z)$

$A' \rightarrow (x, y+\Delta y, z)$

$$F_y(A') = F_y(A) + \frac{\partial F_y}{\partial y} \Delta y + \text{h.o.t}$$

— Taylor expansion.

$$\Rightarrow \Phi_y^{\text{Net}} = \left(\frac{\partial F_y}{\partial y}\right) \Delta x \, \Delta y \, \Delta z$$

$$= \left(\frac{\partial F_y}{\partial y}\right) \Delta V$$

similarly the other two directions.

Hence the net flux

$$\underset{\substack{\uparrow \\ \text{surface}}}{\Phi} = \underset{\substack{\uparrow \\ \text{volume}}}{\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) \Delta V}$$

over an infinitismal cartesian volume

$$\sum_{\text{all sides}} \vec{F} \cdot \hat{n} \, \Delta S = \left(\vec{\nabla} \cdot \vec{F}\right) \Delta V$$

An arbitrarily shaped ballon can always be decomposed into infinitismal volume.

S

Summing up over such a volume

$$\boxed{\oint_S \vec{F}\cdot\hat{n}\, ds = \int_V \left(\vec{\nabla}\cdot\vec{F}\right) dv}$$ Gauss's theorem.

because the flux through all the internal surfaces cancel each other on the left.

$$\vec{\nabla} \equiv \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right)$$

comment:

(i) although we used cartesian boxes the end result is coordinate system independent.

(ii) $\vec{F}(x,y,z)$ need to be smooth enough that it's first derivative exists.

1.13 Back to charge conservation:

$$\int_V (\partial_t \rho)\, dV = \oint_S \vec{J} . \hat{n}\, ds$$

$$= \int_V (\vec{\nabla} . \vec{J})\, dV$$

where V can be any volume.

$$\Rightarrow \quad \boxed{\partial_t \rho + \nabla . \vec{J} = 0}$$

comment

(a) General form of all conservation laws.

1.14

Electric field

$$\oint_S \vec{E}\cdot\hat{n}\, ds = \frac{1}{\epsilon_0} q_{enc}$$

$$= \frac{1}{\epsilon_0}\int_V \rho\, dV$$

continium description

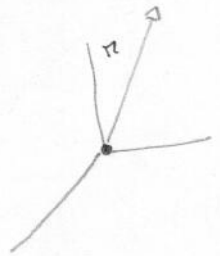
Flux theorem

$$\int_V \left(\vec{\nabla}\cdot\vec{E}\right) dV = \frac{1}{\epsilon_0}\int_V \rho\, dV$$

$$\Rightarrow \quad \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}}$$

1.15

"Point charge"

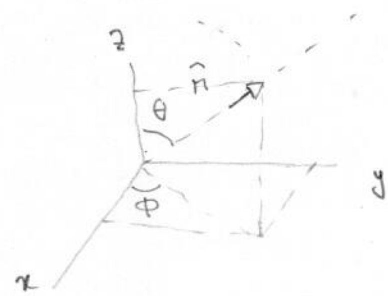


$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{\hat{i}x + \hat{j}y + \hat{k}z}{\left(x^2 + y^2 + z^2\right)^{3/2}}$$

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$$



$$\hat{r} = \hat{z}\cos\theta + \hat{x}\sin\theta\cos\phi$$

$\hat{y}$

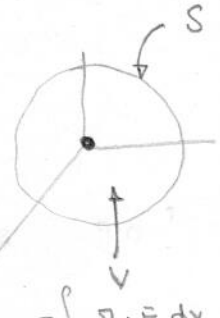
$$\vec{\nabla}\cdot\vec{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{x\,(2x)}{(x^2+y^2+z^2)^{5/2}}\,\frac{3}{2}\right.$$

$$+ \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{y\,(2y)}{(x^2+y^2+z^2)^{5/2}}\,\frac{3}{2}$$

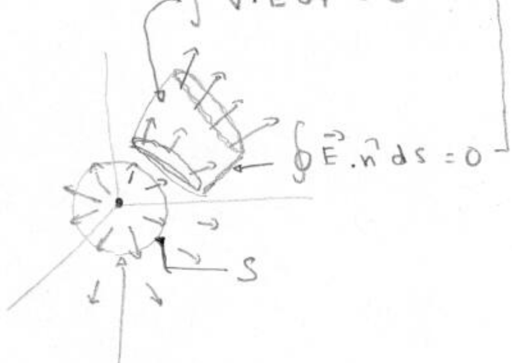
$$\left. + \frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{z\,(2z)}{(x^2+y^2+z^2)^{5/2}}\,\frac{3}{2}\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3\,(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$= 0$$



$$\left.\begin{aligned} \oint_S \vec{E}\cdot\vec{n}\,ds &= \frac{q}{\epsilon_0} \\ \int_V (\bar{\nabla}\cdot\bar{E})\,dV &= 0 \quad ? \end{aligned}\right\}$$



But not for a S that encloses origin

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\,\delta^3(\vec{r})$$

such that $\displaystyle\int_V \delta^3(\vec{r})\,dV = 4\pi$